

Contributions to the ergodic theory of hyperbolic flows: unique ergodicity for quasi-invariant measures and equilibrium states for the time-one map

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Abstract

We consider the horocyclic flow corresponding to a (topologically mixing) Anosov flow or diffeomorphism, and establish the uniqueness of transverse quasi-invariant measures with Hölder Jacobians. In the same setting, we give a precise characterization of the equilibrium states of the hyperbolic system, showing that existence of a family of Radon measures on the horocyclic foliation such that any probability (invariant or not) having conditionals given by this family, necessarily is the unique equilibrium state of the system.

1 Introduction and Main Results

In this article we study aspects related to the thermodynamic formalism for systems having some hyperbolicity, as the time-one map of an hyperbolic flow, or an Anosov diffeomorphism. This is one of the most important pieces in smooth ergodic theory, encompassing several examples and general theorems. Although this part of the theory is fairly complete, the results discussed here are not present in the literature (in this generality, as far as we know), and therefore the present project has a two-fold purpose; on the one hand, introducing the aforementioned results, and on the other serve a (crude) blueprint for future research in more general cases.

We will rely on our previous work [1], and in fact the present manuscript can be seen as a complement to this other one. Nevertheless, we will give a self-contained presentation and no a priori knowledge of this other work is necessary.

By an hyperbolic flow we mean a (C^1) flow $\Phi = \{\Phi_t : M \rightarrow M\}_{t \in \mathbb{R}}$ of a closed manifold satisfying: there exist a Riemannian metric on M , a splitting of the form $TM = E^s \oplus E^c \oplus E^u$, and $0 < \lambda < 1$ such that the following conditions are true.

1. E^c is generated by the vector field tangent to the flow.
2. $D_x \Phi_t(E^{s/u}(x)) = E^{s/u}(\Phi_t(x))$, for all $t \in \mathbb{R}$, for every $x \in M$.
3. $\sup_{x \in M} \|D_x \Phi_t|E^s(x)\|, \sup_{x \in M} \|D_x \Phi_{-t}|E^u(x)\| \leq \lambda^t$, for all $t \geq 0$.

We call E^s, E^u the strong stable and strong unstable bundle, respectively, and E^c will be referred as the center bundle. A hyperbolic flow is said to be of codimension-one if $\dim E^u = 1$. The prototypical example of codimension-one hyperbolic flow is the geodesic flow corresponding to a (closed) surface of negative sectional curvature, but even in dimension three there are many other Anosov flows which are not of this type. We refer the reader to the treatise [2] for an up to date introduction and examples.

The concept of hyperbolic diffeomorphism is analogous; we say that a diffeomorphism $f : M \rightarrow M$ of an closed manifold M is hyperbolic (or Anosov) if there exist a Riemannian metric on M , a splitting of the form $TM = E^s \oplus E^u$, and $0 < \lambda < 1$ such that the following conditions are true.

1. $D_x f(E^{s/u}(x)) = E^{s/u}(f(x))$, for every $x \in M$.

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$$2. \sup_{x \in M} \|D_x f|E^s(x)\|, \sup_{x \in M} \|D_x f|E^u(x)\| \leq \lambda.$$

Remark 1.1. For the rest of the article we will work only with (codimension-one) hyperbolic flows, leaving the direct modifications for the diffeomorphism case to the reader.

Now fix a codimension-one hyperbolic flow; by the classical Hadamard-Perron's stable manifold theorem, it follows in this setting that E^u integrates, provided that it is orientable, to a (in general, only transversally continuous) flow $\Phi^u = \{\Phi_t^u : M \rightarrow M\}_{t \in \mathbb{R}}$, called the horocyclic flow. It turns out that if we further assume transitivity of Φ , meaning the existence of a dense orbit, there are only two possibilities [3]; either

1. M is a fiber bundle over S^1 , with fibers transverse to the flow lines (the suspension case), or
2. Φ^u is minimal, that is, every orbit of Φ^u is dense in M .

The geodesic flow corresponding to a negatively curved surface falls into this last category.

Convention. For the rest of this article we assume that the unstable bundle of any considered hyperbolic flow is orientable. Modulo a two-fold covering, this not a significant loss of generality.

We then have the following.

Theorem 1.1. *If Φ is a (transitive) codimension one Anosov flow that is not a suspension, then Φ^u is uniquely ergodic. That is, there exists only one (probability) measure invariant under Φ^u .*

Transverse and invariant measures are in one-to-one correspondence for flows, so the above theorem says that there is only one transverse invariant measure for the flow foliation. The previous theorem is an important result originally due to H. Furstenberg [4] for the geodesic flow corresponding to hyperbolic surfaces, and later generalized by B. Marcus [5] to the variable curvature case, and in fact to arbitrary codimension-one Anosov flows (that are not a suspension). The most general result is due to R. Bowen and B. Marcus, where they do not assume one-dimensionality of the unstable bundle [6].

In this article we generalize the above result. Let us introduce some definitions.

Definition 1.1. *Let Ψ be a flow on M . A (local) transversal to the flow is a sub-manifold $D \subset M$ that is transverse to the flow direction. A pairwise disjoint family $\mathcal{D} = \{D\}_D$ transversal is a global transversal if given $x \in M$ there exist $t \in \mathbb{R}$ and $D \in \mathcal{D}$ such that $\Phi_t(x) \in D$. The global transversal \mathcal{D} is full if for every $x \in M$ there exists $D \in \mathcal{D}$ with $x \in D$.*

A transverse measure for Φ is a family of measures $\nu = \{\nu_D : D \in \mathcal{D}\}$ where \mathcal{D} is a global transversal to Φ and for each $D \in \mathcal{D}$, ν_D is a Radon measure on D : \mathcal{D} is the support of ν . We will assume that the support of the transverse measures considered is full¹

If $\nu = \{\nu_D : D \in \mathcal{D}\}$ is a transverse measure to Φ we also write $\nu = \{\nu_x\}_{x \in M}$ where $\nu_x = \nu_D, x \in D$.

Definition 1.2. *Let Ψ be a flow on M and $\{\nu_x\}_x$ be a transverse measure for Ψ . We say that $\{\nu_x\}_x$ is quasi-invariant if there exist a family of positive functions $\text{Jac}_\nu = \{\text{Jac}_{x_0, y_0} : M \rightarrow \mathbb{R}_{>0} : y_0 = \Psi_{t_0}(x_0), x_0 \in M, t_0 \in \mathbb{R}\}$ and $C > 0$ such that*

$$\Psi_{-t_0} \nu_{y_0} = C \text{Jac}_{x_0, y_0} \nu_{x_0}.$$

In this case Jac_ν is the Jacobian of the quasi-invariant measure, whereas C is the associated normalization constant.

Remark 1.2. *Note that if Jac_ν is a Jacobian, then it defines a multiplicative cocycle over the flow.*

An invariant (transverse) measure corresponds to $\text{Jac}_\nu \equiv 1$. We will be interested in quasi-invariant transverse measures for Φ^u having well-behaved Jacobian, and having some relation with the dynamics Φ . As a convenient family of transversals for the horocyclic flow we consider the leaves of the center stable foliation $\mathcal{W}^{cs} = \{W^{cs}(x)\}_{x \in M}$, which are tangent to $E^{cs} = E^c \oplus E^s$. This choice of transversals is implicit in what follows, and in particular transverse measures to Φ^u are assumed to be supported in center stable leaves. The existence and basic properties of invariant foliations associated to Φ are recalled in section 2.

We first state the following.

¹This is not loss of generality for our purposes.

Proposition 1.2 (Proposition 2.10 in [1]). *Suppose that $\mu^{cs} = \{\mu_x^{cs}\}$ is a transverse measure for Φ^u such that $\forall x \in M$*

$$\Phi_{-1}\mu_{\Phi_1(x)}^{cs} = Ch\mu_x^{cs},$$

where $C > 0$ and $h : M \rightarrow \mathbb{R}_+$ is Hölder. Then μ^{cs} is Φ^u -quasi invariant, with Jacobian

$$\text{Jac}_{x_0, y_0}^u(x) = \prod_{j=1}^{\infty} \frac{h(\Phi_{-j} \circ \Phi_{t(x)}^u x)}{h(\Phi_{-j}(x))}.$$

where for $x \in W^{cs}(x_0)$, $\Phi_{t(x)}^u(x) \in W^{cs}(y_0)$.

Consider the set of Hölder multiplicative cocycles

$$\mathcal{C}_{o-c}(\Phi) = \{h : M \rightarrow \mathbb{R}_+ : h(x) = e^{\int_0^1 \varphi(\Phi_t x) dt} \text{ for some Hölder function } \varphi\}. \quad (1)$$

This is a natural class of functions for Φ , related to its equilibrium states; see below. In the next section we recall the by now classical fact that any equilibrium state of the flow Φ corresponding to a Hölder potential φ determines $h \in \mathcal{C}_{o-c}(\Phi)$ and a Φ^u -transverse quasi invariant measure μ^{cs} satisfying the hypotheses of the previous proposition. Given $h \in \mathcal{C}_{o-c}(\Phi)$ define the family of functions

$$\mathfrak{h} = \{H_{x_0, y_0} : x_0, y_0 \in M, y_0 \in W^u(x_0)\} \quad (2)$$

with

$$H_{x_0, y_0}(x) = \prod_{j=1}^{\infty} \frac{h(\Phi_{-j} \circ \Phi_{t(x)}^u x)}{h(\Phi_{-j}(x))}, \quad x \in W^{cs}(x_0), \Phi_{t(x)}^u(x) \in W^{cs}(y_0). \quad (3)$$

Our first main result is as follows.

Theorem A. *Let Φ be a codimension-one Anosov flow of class C^2 that is not a suspension, and consider $h \in \mathcal{C}_{o-c}(\Phi)$. Assume that the unstable bundle of Φ is orientable and denote by Φ^u the induced horocyclic flow. Then there exists $\mu^{cs} = \{\mu_x^{cs}\}_x$ a transverse measure for Φ^u such that μ^{cs} is the unique quasi-invariant measure with Jacobian given by the family \mathfrak{h} determined by h .*

Given a flow $\Psi = \{\psi_t\}_t$ on M and J a non-negative multiplicative cocycle over Ψ , a (Borel) measure μ is said to be J -conformal if $\psi_t^{-1}\mu = J_t \cdot \mu$; the related definition for diffeomorphism is similar. The notion was introduced by Patterson [7] in the context of limit sets for Fuchsian groups, and has an important role in geometry and ergodic theory. The reader is directed to the book [8] for an extensive discussion of conformal measures in ergodic theory, and to the survey [9] for a discussion of applications in geometry, in particular of the Patterson-Sullivan theory.

We single out the following remarkable existence and uniqueness result for conformal measures due to A. Douady and J-C. Yoccoz [10].

Theorem 1.3. *Let $f : \mathbb{T} \rightarrow \mathbb{T}$ be a C^2 diffeomorphism of the circle with irrational rotation number. Then for every $s \in \mathbb{R}$ there exists a unique $s \cdot Df$ -conformal measure.*

The following is a re-writing of Theorem A, which in view of the importance of the concept of conformal measures we isolate it as a corollary.

Corollary A. *Under the same hypotheses of Theorem A, the horocyclic flow Φ^u has a unique \mathfrak{h} -conformal measure.*

The previous two results are evidence that parabolic systems may not have many more than one conformal measure; we can ask

Question. *If Φ is a minimal parabolic flow and J is non-negative multiplicative cocycle, does there exist a unique J -conformal measure.*

This question is particularly interesting for unipotent flows, in view of the very general rigidity results due to Ratner [11], and others derived from her work.

Equilibrium States. The proof of the above Theorem A relies on the study of equilibrium states for the time-one map of the corresponding hyperbolic flow. Let us recall some basic definitions and refer the reader to [1] and references therein for a more through discussion.

For a continuous endomorphism of a compact metric space $f : M \rightarrow M$ we denote by $\mathcal{P}r_f(M)$ the set of f -invariant probability measures, equipped with the ω^* -topology. Given a continuous real valued map $\varphi : M \rightarrow \mathbb{R}$ (the potential), the topological pressure of the system is given as [12]

$$P_{top}(\varphi) = \sup_{\nu \in \mathcal{P}r_f(M)} \{h_\nu(f) + \int \varphi d\nu\} \quad (4)$$

where $h_\nu(f)$ denotes the metric entropy of f with respect to ν . We denote

$$\mathcal{E}q_f = \{\mu \in \mathcal{P}r_f(M) : P_{top}(\varphi) = h_\mu(f) + \int \varphi d\mu\}; \quad (5)$$

elements in $\mathcal{E}q_f$ are called *equilibrium states* (for the system (f, φ)).

Similarly, if $\Phi = (\Phi_t)_t$ is a flow we denote $\mathcal{P}r_\Phi(M) = \cap_t \mathcal{P}r_{\Phi_t}(M)$ the set of its invariant measures; the topological pressure of the system (Φ, φ) is given as

$$P_{top}(\Phi, \varphi) = \sup_{\nu \in \mathcal{P}r_\Phi(M)} \{h_\nu(\Phi_1) + \int \varphi d\nu\}, \quad (6)$$

and we define $\mathcal{E}q(\Phi, \varphi)$ analogously. One can show (cf. proposition 6.2 in [1]) that for a (mixing) Anosov flow one has $\mathcal{E}q(\Phi, \varphi) = \mathcal{E}q(\Phi_1, \tilde{\varphi})$ where

$$\tilde{\varphi}(x) = \int_0^1 \varphi(\Phi_t x) dt.$$

The next theorem is a central piece in smooth ergodic theory.

Theorem ([13, 14, 15]). *Let Φ be a transitive Anosov flow of class C^1 and $\varphi : M \rightarrow \mathbb{R}$ be a Hölder function, Then $\mathcal{E}q(\Phi, \varphi)$ consists of a unique element $m_{\Phi, \varphi}$.*

Here we are able to give the following strong characterization of $m_{\Phi, \varphi}$ in terms of conditional measures along unstables. Let us recall that given a probability measure m on M a measurable partition (in the sense of Rokhlin [16]) ξ is said to be

- increasing if $\Phi_1 \xi < \xi$ (i.e. for m -a.e.(x) the atom $\Phi_1(\xi(x))$ consists of atoms of ξ);
- subordinated to \mathcal{W}^u is for m -a.e.(x) it holds $\xi(x) \subset W^u(x)$;
- an SLY² partition if it is increasing, subordinated to \mathcal{W}^u , and for m -a.e.(x) the atom $\xi(x)$ contains a relative neighborhood of x in $W^u(x)$. See

It is by now classical (and holds in much more generality than the case we are considering here) that for Anosov flows there exist SLY partitions or arbitrarily small mesh. See for example [17].

We can prove the following.

Theorem B. *Assume the same hypotheses of Theorem A. Then there exists a family of measures $\{\nu_x^u\}_{x \in M}$, where each ν_x^u is a Radon measure on $W^u(x)$, such that for any probability m on M (not necessarily invariant under any Φ_t) whose conditionals in some m SLY partition ξ are of the form*

$$m_x^\xi = \frac{\nu_x^u}{\nu_x^u(\xi(x))},$$

necessarily satisfies $m = m_{\Phi, \varphi}$.

It is worth to emphasize that above we are not requiring the measure to be invariant (or quasi-invariant), and we do not assume any information in the transverse direction; Theorem B evidences a strong rigidity property of equilibrium states for hyperbolic systems.

²By Sinai, Strelcyn, Ledrappier and Young

Comparison with existing literature Theorem A was established originally by M. Babillot and F. Ledrappier [18] in the case of (Abelian covers of) the geodesic flow in a hyperbolic closed manifold, and later proven by a more geometrical argument by B. Schapira [19] in the same setting. Our method has some similarities with Schapira’s, but is less dependent on non-trivial geometrical considerations for the geodesic flow, and is more direct. This allows us to consider general (codimension one) Anosov flows and diffeomorphisms. Of course, the tailored arguments for the geodesic flow give additional information in that case, and are generalizable for some non-compact hyperbolic manifolds; see [20], [21]. Applicability of the ideas in this paper to the non-compact case remains to be investigated.

The case of diffeomorphisms (Anosov) can be deduced from the work of Babillot and Ledrappier above cited, using the symbolic model of the map to translate the problem to subshifts of finite type, and then applying the powerful tools of spectral theory of transfer operators in this setting. For this, one needs to consider only the positive part of the shift (that is, the unstable sets) since the behavior of invariant measures on the other “transverse” direction (stable) is determined by the former. This is a particularity of the symbolic model, and has been exploited extensively. For example, in [series1980] C. Series obtains the analogue of Theorem A for the lifted foliations to the symbolic model, and classifies the Borel equivalence relations induced by these. This was pushed further in [22], still in the symbolic setting. To recover the same result for the diffeomorphism (uniqueness of the quasi-invariant transverse measure) one needs to control an additional direction; this is non-trivial but could be done, particularly in cases where the measure is known to be well behaved (as the entropy maximizing measure or the SRB).

The case of flows is different, since typically (for example, for mixing Anosov flows) there is no possible symbolic model for the time- t maps, and hence one cannot reduce their study to subshifts. What one has is a suspension of a subshift covering the flow, but this suspension is via a non-cohomologous to constant function, which depends on some arbitrary choices, and therefore cannot be used to give much information about the individual time- t maps. See Bowen’s article [23].

We follow here a completely different approach, without reducing the problem to the symbolic flow, and this allows us to control the transverse direction as well. We remark also that in Babillot and Ledrappier’s work (and also, Schapira’s) the transverse direction is controlled using the rigid geometry of the map, and that’s probably the main reason why their method cannot also be applied to non-symmetric Anosov flows.

Finally, let us point out that Theorem B is new even for (linear!) Anosov diffeomorphisms, as we are not requiring invariance of the measure.

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2 Measures along strong unstable leaves and characterization of equilibrium states

In this part we give the preliminaries necessary for the proof Theorem A and Theorem B.

From now on $\Phi = (\Phi_t : M \rightarrow M)_{t \in \mathbb{R}}$ denotes a transitive hyperbolic flow of codimension-one that is not a suspension, with associated invariant decomposition $TM = E^s \oplus E^c \oplus E^u$, with E^u tangent to the flow lines of the horocyclic flow Φ^u . The bundles $E^s, E^u, E^{cs} = E^c \oplus E^s, E^{cu} = E^c \oplus E^u$ are all integrable to flow-invariant foliations \mathcal{F}^* , where the superscript $*$ coincides with the one of the bundle that it integrates. In particular, \mathcal{W}^u is the foliation by orbits of Φ^u . Each leaf of $\mathcal{W}^{cs}, \mathcal{W}^{cu}$ is saturated by leaves of $\mathcal{W}^s, \mathcal{W}^c$ and $\mathcal{W}^u, \mathcal{W}^c$, respectively.

Notation: If $A \subset M, \epsilon > 0$ and $\mathcal{F}^* = \{W^*(x)\}_{x \in M}$ is one of the invariant foliations, let

$$W^*(A, \epsilon) := \{y \in M : \exists x \in A / d_{\mathcal{F}^*}(x, y) < \epsilon\},$$

where $d_{\mathcal{F}^*}$ is the intrinsic distance in the corresponding leaf. If $A = \{x\}$ we write $W^*(x, \epsilon) = W^*(\{x\}, \epsilon)$.

Since the foliations $\mathcal{W}^u, \mathcal{W}^{cs}$ are transverse, there exists some $c_{geo} > 0$ such that

$$x, y \in M, d(x, y) < \epsilon \leq c_{geo} \Rightarrow \#W^u(x, 2\epsilon) \cap W^{cs}(y, 2\epsilon) = 1.$$

For a submanifold $L \subset M$ denote $\mathcal{R}ad(L)$ the set of Radon measures (for the induced topology) on L , and for \mathcal{F}^* consider

$$\begin{aligned} \text{Rad}(\mathcal{F}^*) &:= \bigsqcup_{L \in \mathcal{F}^*} \mathcal{R}ad(L) \\ \text{Meas}(\mathcal{F}^*) &:= \{\nu : M \rightarrow \text{Rad} : \nu_x := \nu(x) \in \mathcal{R}ad(W^*(x)) \forall x \in M\}. \end{aligned}$$

We have the following.

Proposition 2.1 (Proposition 6.1 in [1]). *There exists $\mu^u \in \text{Meas}(\mathcal{W}^u)$ satisfying:*

- a) μ_x^u is non-atomic and $\text{supp}(\mu_x^u) = W^u(x)$.
- b) $\forall t \in \mathbb{R}, x \in M$, it holds

$$\Phi_{-t}\mu_{\Phi_t(x)}^u = e^{P_{\text{top}}(\varphi)t - \int_0^t \varphi(\Phi_s(\cdot))ds} \mu_x^u.$$

- c) *The map $x \rightarrow \mu_x^u$ is weakly continuous, meaning: given $y \in W^{cs}(x, c_{geo})$ and $\text{hol}_{x,y}^s : W^u(y) \rightarrow W^u(x)$ the locally defined Poincaré map determined by the center unstable foliation, it follows that for any $A \subset W^u(x)$ relatively open and pre-compact it holds*

$$\mu_y^u(\text{hol}_{x,y}^s(A)) \xrightarrow{y \rightarrow x} \mu_x^u(A).$$

If moreover $A \subset W^u(x, c_{geo})$, then the convergence is uniform in x .

With this we were able to establish the next theorem.

Theorem 2.2 (Theorem 6.3 in [1]). *If Φ is C^2 Anosov flow with minimal unstable foliation, then the family μ^u of the previous theorem satisfies: if $m \in \mathcal{P}^r(M)$ satisfies*

1. m is Φ_t -invariant for some $t \neq 0$, and
2. m has conditionals absolutely continuous with respect to μ^u ,

then $m = m_{\Phi, \varphi}$.

From now on we denote $f = \Phi_1$ and fix $\varphi : M \rightarrow \mathbb{R}$ Hölder potential together with its corresponding measures μ^u given in Proposition 2.1. We denote

$$h_\varphi(x) = e^{\tilde{\varphi}(x)},$$

and given $x \in M$ define $\Delta_x^u : W^u(x) \rightarrow \mathbb{R}_{>0}$,

$$y \in W^u(x) \Rightarrow \Delta_x^u(y) := \prod_{k=1}^{\infty} \frac{h_\varphi(f^{-k}y)}{h_\varphi(f^{-k}x)}. \quad (7)$$

Since $\tilde{\varphi}$ is Hölder it follows that Δ_x^u is continuous, and converges to one as $y \mapsto x$.

Definition 2.1. For $x \in M$ let $\nu_x^u = \Delta_x^u \mu_x^u$.

Clearly $\nu_x^u \in \mathcal{R}\text{-}a\text{-}d(W^u(x))$ and satisfies the following properties.

- a) If $y \in W^u(x)$ we have $\nu_y^u = c(y, x) \cdot \nu_x^u$ for some constant $c(y, x) = \Delta_y^u(x) > 0$, hence $\{\nu_y^u\}_{y \in W^u(x)}$ defines a projective class of measures $[\nu_x^u]$ in $W^u(x)$.
- b) $f^{-1}\nu_{fx}^u = e^{P_{\text{top}}(\varphi) - \tilde{\varphi}(x)} \nu_x^u$.

Definition 2.2. A family of measures $\zeta^{cs} \in \text{Meas}(\mathcal{W}^{cs})$ is said to be compatible with (φ, μ^u) if its quasi-invariant for Φ^u with Jacobian given by the family h_{ν_φ} (cf. Eq. (2)).

We proved before (Section 3.2 of [1]) that given ζ^{cs} compatible with (φ, μ^u) one can construct a probability measure m on M that is given locally as follows. For $W = W^{cs}(x_0, \epsilon)$, $0 < \epsilon \leq c_{geo}$,

$$m \approx \int_W \nu_x^u d\zeta_{x_0}^{cs}(x).$$

Compatibility is used to show that the definition does not depend on the chosen center stable disc, and therefore one can glue these local measures into m .

Remark 2.1. Moreover, if ζ^{cs} is a quasi-invariant measure for f as in Proposition 1.2, then the resulting m is an equilibrium state for $(f, \tilde{\varphi})$, and therefore the unique equilibrium state for the system (Φ, φ) . This does not require codimension-one and was noted in our previous article.

We point out that there exists a family $\mu^{cs} \in \text{Meas}(\mathcal{W}^{cs})$ satisfying the quasi-invariance condition refereed in the previous remark. The equilibrium state $m_{\Phi, \varphi}$ can be constructed using this family.

3 Unique ergodicity for the conditional measures

Let $c_{int} > 0$ be much smaller than c_{geo} , and for $x \in M$ define

$$I_x := \Phi_{[-c_{int}, c_{int}]^u}(x). \quad (8)$$

The set I_x will be called a u -interval. Since the conditionals ν_x^u are non-atomic (cf. Proposition 2.1), we deduce:

Lemma 3.1. *It holds $\forall x \in M, \nu_x^u(\partial I_x) = 0$.*

Let $\mathcal{I} = \{I_x\}_{x \in M}$ and observe that the family \mathcal{I} has the following continuity property:

$$I_y \xrightarrow[y \rightarrow x]{\text{Haus}} I_x$$

where the above convergence is in the Hausdorff topology. We also have the following.

Lemma 3.2. *The functions $x \rightarrow \mu_x^u(I_x)$ and $x \rightarrow \nu_x^u(I_x)$ are continuous.*

Proof. By Eq. (7) it suffices to prove the first part, which follows by c) of Proposition 2.1, the lemma above, and Alexandrov's theorem. \blacksquare

Recall that $f = \Phi_1$ denotes the time-one map of a (topologically mixing) codimension-one Anosov flow. For $\mathfrak{h} \in \mathcal{C}(M)$ we define the operator

$$R_n \mathfrak{h}(x) = \frac{1}{\nu_{f^n(I_x)}^u(f^n(I_x))} \int_{f^n(I_x)} \mathfrak{h}(t) d\nu_{f^n(x)}^u(t) = \frac{1}{\nu_x^u(I_x)} \int_{I_x} \mathfrak{h} \circ f^n(t) d\nu_x^u(t). \quad (9)$$

From now on we fix m_φ an equilibrium state for $(f, \tilde{\varphi})$ constructed as explained in the previous section.

Notation. We write $W_{loc}^*(x) = W^\sigma(x, c_{geo})$. If B is foliation box of diameter less than c_{geo} , the connected component of $W_{loc}^*(x) \cap B$ that contains x is denoted by $W^*(x, B)$.

We start by establishing the following.

Proposition 3.3. *For every $\mathfrak{h} \in \mathcal{C}(M)$ the family $\{R_n \mathfrak{h}\}_n$ is equicontinuous and uniformly bounded.*

Proof. As the family $\{R_n \mathfrak{h}\}_n$ is clearly bounded by $\|\mathfrak{h}\|_{C^0}$ we only need to show that it is equicontinuous. To this end, given two points x, y with $d(x, y) < c_{geo}$ we can choose z and w so that $z \in W_{loc}^u(x) \cap W_{loc}^{cs}(y)$ and $w \in W_{loc}^c(z) \cap W_{loc}^s(y)$. See Fig. 1 below. To show equicontinuity we will make comparisons between x and z , z and w and w and y .

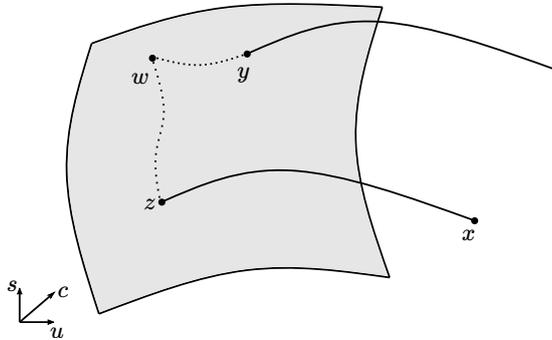


Figure 1: Diagram for the proof.

Since $z \in W^u(x)$ we have $\nu_x^u = c(x, z)\nu_z^u$, where $c(x, z)$ converges uniformly to one as z approaches x (cf. the remark after the definition of the measures ν_x^u). Therefore

$$\begin{aligned} |R_n \mathfrak{h}(x) - R_n \mathfrak{h}(z)| &= \left| \frac{1}{\nu_x^u(I_x)} \int_{I_x} \mathfrak{h} \circ f^n(t) d\nu_x^u(t) - \frac{1}{\nu_x^u(I_z)} \int_{I_z} \mathfrak{h} \circ f^n(t) d\nu_x^u(t) \right| \\ &\leq \|\mathfrak{h}\|_{C^0} \int \left| \frac{\mathbb{1}_{I_x}(t)}{\nu_x^u(I_x)} - \frac{\mathbb{1}_{I_z}(t)}{\nu_x^u(I_z)} \right| d\nu_x^u(t) \leq \frac{2\|\mathfrak{h}\|_{C^0}}{\nu_x^u(I_x)} \cdot \nu_x^u(I_x \Delta I_z) \end{aligned}$$

and the later term can be taken arbitrarily small if z is close enough to x , by Lemmas 3.1 and 3.2.

The comparison between w and y follows similar lines, using Proposition 2.1. Finally, for the comparison between z and w we observe that the center holonomy (i.e. the flow) sends ν_z^u to $g_{z,w}\nu_w^u$, where $g_{z,w}$ is a continuous positive function that converges uniformly to one as $z \mapsto w$. Therefore, we get

$$\begin{aligned} &\left| \frac{1}{\nu_x^u(I_z)} \int_{I_z} \mathfrak{h} \circ f^n(t) d\nu_z^u(t) - \frac{1}{\nu_y^u(I_w)} \int_{I_w} \mathfrak{h} \circ f^n(t) d\nu_w^u(t) \right| \\ &= \left| \frac{1}{\nu_z^u(I_z)} \int \left(\mathbb{1}_{I_z}(t) \mathfrak{h} \circ f^n(t) - \frac{\nu_z^u(I_z)}{\nu_w^u(I_w)} \mathbb{1}_{\text{hol}_{z,w}^c(I_z)}(t) \mathfrak{h} \circ f^n(\text{hol}^c(t)) \cdot g_{z,w}(t) \right) d\nu_w^u(t) \right| \end{aligned}$$

As $z \mapsto w$ we have $\frac{\nu_z^u(I_z)}{\nu_w^u(I_w)} \rightrightarrows 1$ as well. Since f is part of the flow, we conclude that $|\mathfrak{h} \circ f^n(\text{hol}^c(t)) - \mathfrak{h} \circ f^n(t)|$ can be taken uniformly small for z close to w , independently of n . Altogether we have shown that for every n , $|R_n \mathfrak{h}(z) - R_n \mathfrak{h}(w)|$ is small whenever $w \in W^c(z)$ is close to z , and this concludes the proof of equicontinuity. \blacksquare

3.1 SOT convergence of $(R_n)_n$.

Now we will show the central part of our argument: the family of positive operators $(R_n : \mathcal{C}(M) \rightarrow \mathcal{C}(M))_n$ converges in the strong operator topology to some multiple of the identity. More precisely, there exists some probability measure p^u such that

$$\mathfrak{h} \in \mathcal{C}(M) \Rightarrow \|R_n \mathfrak{h} - \int \mathfrak{h} dp^u\|_{C^0} \xrightarrow{n \rightarrow \infty} 0.$$

Due to Riesz-Markov representation theorem, this amounts to establishing the following.

Theorem 3.4. *For every $\mathfrak{h} \in \mathcal{C}(M)$ the sequence $(R_n \mathfrak{h})_n$ converges uniformly to a constant a .*

We first determine the value a . Define Ω as the set of all sequences $\{\Theta_n^m\}_{m,n \geq 0}$ of functions $\Theta_n^m : M \rightarrow \text{Rad}(\mathcal{W}^u)$ satisfying

1. $\Theta_{n,x}^m := \Theta_n^{m,n}(x)$ is a probability measure on $f^m(I_x)$, and
2. given $\epsilon > 0$ there exists $N_\epsilon > 0$ such that for $n, m \geq N_\epsilon$ it holds for every $x \in M$

$$R_{n+m} \mathfrak{h}(x) = \int R_n \mathfrak{h}(y) d\Theta_{n,x}^m(y) + E_{n,m}(x) \tag{10}$$

where $|E_{n,m}(x)| < \epsilon$.

Assume for the moment that Ω is non-empty and take $\{\Theta_n^m\}_{n,m \geq 0} \in \Omega$. Define

$$a_n := \inf_x R_n \mathfrak{h}(x)$$

and observe that by (10)

$$a_{n+m} \geq a_n - \sup_x |E_{n,m}(x)|.$$

Let $a := \limsup_n a_n$, $b = \liminf_n a_n$; these are finite numbers in virtue of Proposition 3.3. Given $\epsilon > 0$ take N_ϵ such that $n, m \geq N_\epsilon$ implies $\sup_x |E_{n,m}(x)| < \epsilon$: since any neighborhood of a contains points of (a_n) , by the previous formula we deduce that $|b - a| < 2\epsilon$, and as ϵ is arbitrary we have that $a = b$, therefore

$$a = \lim_n a_n.$$

Now that we have our candidate for constant function, we will proceed as follows: consider $\mathbf{g} \in \mathcal{C}(M)$ an accumulation point of some sub-sequence $(R_{n_k} \mathfrak{h})_k$ (which exists by equicontinuity of the family). Given an open set $U \subset M$, we will find $\Theta \in \mathcal{P}^r(M)$ with $U \subset \text{supp}(\Theta)$ and $\int_U (\mathbf{g} - a) d\Theta = 0$. Since $\mathbf{g}(x) \geq a$ for every x , this will imply the existence of some $x_U \in U$ such that $g(x_U) = a$ on U . As U can be chosen in a basis of the topology on M , we will deduce that $g = a$, thus showing $\lim_n \|R_n \mathfrak{h} - a\|_{C^0} = 0$.

The remainder of this part is devoted to establishing this result.

Take $U \subset M$ open box of the form $U \approx W^c(W^s(x_0, \delta), \delta) \times W^u(x_0, \delta)$ with δ much smaller than c_{geo} .

Notation. For $0 < \rho \ll c_{geo}$ and $S \subset M$ we denote by $\partial_\rho S$ the ρ -neighborhood of the boundary of S . If $U \approx W^c(W^u(x_0, \delta), \delta) \times W^s(x_0, \delta)$ is a box then its cu -boundary of size ρ is the set $\partial_\rho^{cu} U$ corresponding to $W^c(W^u(x_0, \delta), \delta) \times \partial_\rho W^s(x_0, \delta)$.

Lemma 3.5. *If $m_\varphi(\partial U) = 0$ there exists $C_U > 0$ such that for every $x \in M$*

$$\liminf_{m \rightarrow \infty} \frac{\nu_{f^m x}(U \cap f^m(I_x))}{\nu_{f^m x}(f^m(I_x))} > C_U > 0.$$

Remark 3.1. *Since (f, m_φ) is a Kolmogorov system, the above inequality holds for m_φ -a.e.(x), however for our purposes it is important to establish the validity for all $x \in M$.*

Proof. Start observing that $\frac{\nu_{f^m x}(U \cap f^m(I_x))}{\nu_{f^m x}(f^m(I_x))} = \frac{\nu_x(f^{-m}U \cap (I_x))}{\nu_x(I_x)}$.

Assume by means of contradiction that the conclusion is not true. Since $\inf_x \nu_x^u(I_x) > 0$ we have that for every $\tau > 0$ there exists x_τ and n_τ such that for every $n \geq n_\tau$ it holds

$$\nu_{x_\tau}(I_{x_\tau} \cap f^{-n}U) \leq \tau.$$

Choose $\rho > 0$ so that $m_\varphi(\partial_\rho^{cu} U)$ is much smaller than τ , set $d_U = \text{diam } U$ and define

$$B_x = \bigcup_{y \in W^c(I_x, 5d_U)} I_y$$

We want to compare the measures $m_\varphi(B_x \cap f^{-n}U)$ and $\nu_x^u(I_x \cap f^{-n}U)$. Divide the connected components of $B_x \cap f^{-n}U$ into two types, Good_x^n and Bad_x^n where $Z \in \text{Good}_x^n \Leftrightarrow$ for every $z \in Z$ it holds

1. $f^n(W^s(z, Z)) \subset W^s(f^n z, U)$;
2. $f^n(W^u(z, Z)) \supset W^u(f^n z, U)$.

Observe that as f preserves center lengths the second condition above implies

3. $f^n(W^{cu}(z, Z)) \supset W^{cu}(f^n z, U)$

It follows that if $Z \in \text{Bad}_x^n$ then either

1. $Z \subset \partial_\rho^{cu} U$, or
2. $f^n(I_x \cap f^{-n}U \cap Z) \not\supset W^u(f^n z, U)$.

Observe that we can write,

$$\frac{m_\varphi(f^{-n}U \cap B_x)}{\nu_x^u(f^{-n} \cap I_x)} = \frac{m_\varphi(\bigcup_{Z \in \text{Good}_x^n} Z)}{\nu_x^u(f^{-n} \cap I_x)} \left(1 + \frac{m_\varphi(\bigcup_{Z \in \text{Bad}_x^n} Z)}{m_\varphi(\bigcup_{Z \in \text{Good}_x^n} Z)} \right)$$

which by the local product structure of m_φ implies, for some uniform c_{prod} ,

$$\frac{m_\varphi(\bigcup_{Z \in \text{Good}_x^n} Z)}{\nu_x^u(f^{-n} \cap I_x)} \left(1 + \frac{m_\varphi(\partial_\rho^{cu} U) + c_{prod} \cdot \nu_x^u(\partial_{\lambda^{-n} 2c_{int}})}{m_\varphi(\bigcup_{Z \in \text{Good}_x^n} Z)} \right).$$

Using again the local product structure we deduce the existence of $\tilde{d} > 0$ satisfying: for every x there exists m_x so that for $n \geq m_x$,

$$m_\varphi(f^{-n}U \cap B_x) \leq \tilde{d} \cdot \nu_x^u(I_x \cap f^{-n}U).$$

Putting all the pieces together we finally conclude that for every $\epsilon > 0$ there exist $x_\epsilon \in M$ and $n'_\epsilon \in \mathbb{N}$ such that if $n \geq n'_\epsilon$ then

$$m_\varphi(B_{x_\tau} \cap f^{-n}U) < \epsilon.$$

Since the system is mixing, we then deduce that for $n \geq n''_\epsilon$,

$$m_\varphi(B_{x_\tau}) \cdot m_\varphi(U) < 2\epsilon.$$

We will show below that $\inf_x m_\varphi(B_x) > 0$: but then we get that $m_\varphi(U) = 0$, contradicting the fact that m_φ has full support. To finish the proof we establish the following.

Claim: If λ is a probability on M with full support, then $\inf_x \lambda(B_x) > 0$.

Suppose not: then we can find a converging sequence $x_n \xrightarrow{n \rightarrow \infty} x$ such that the corresponding sets B_{x_n} converge to B_x in the Hausdorff topology, and furthermore

$$\lambda(B_{x_n}) \xrightarrow{n \rightarrow \infty} 0.$$

Take $D(x, r) \subset B_x$ open disc of radius r , centered at x , and such that $\lambda(\partial D(x, r)) = 0$: as

$$D(x_n, r) \xrightarrow[n \rightarrow \infty]{\text{Haus}} D(x, r)$$

it follows that

$$\lambda(D(x_n, r)) \xrightarrow{n \rightarrow \infty} \lambda(D(x, r)),$$

and hence $\lambda(D(x, r)) = 0$, contrary to our assumption that λ has full support. ■

We want to sub-divide $f^m(I_x)$ in intervals that are well positioned with respect to the connected components of $f^m(I_x) \cap U$; for this we will use the following lemma (cf. lemma 3.2 in [5]).

Lemma 3.6. *Given $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for every $x \in M$, the set*

$$\{\Phi_{t2c_{int}}^u(x) : t = 0, \dots, n_\epsilon\}$$

is ϵ -dense.

To cover $f^m(I_x)$ we use the following algorithm. Denote $d_U = \text{diam } U$ and choose $0 < \epsilon < \frac{d_U}{5}$. For $x \in M$ consider $\{C_x^{(m)}(l)\}_{l=0}^{o(x,m)}$ the set of connected components of $f^m(I_x) \cap U$, ordered according to the orientation of $f^m(I_x)$; note that since d_U is small with respect to c_{int} , if $y \in C_x^{(m)}(l)$ then $I_y \cap U = C_x^{(m)}(l)$. Choose $y_{x,0}^{(m)} \in C_x^{(m)}(0)$ and for $n \in \mathbb{Z}$ let $y_{x,n}^{(m)} = \Phi_{n2c_{int}}^u(y_{x,0}^{(m)})$. Observe that by applying the above lemma to the family $\{y_{x,n}^{(m)}\}_{n \geq 0}$ we get

$$\forall l, \#\{l \leq n \leq l + 2n_\epsilon : y_{x,n}^{(m)} \in U\} \geq n_\epsilon. \quad (11)$$

After choosing $y_{x,0}^{(m)}$ consider $\bar{k}_x^m = \bar{k}_x^m(y_{x,1}^m), \underline{k}_x^m = \underline{k}_x^m(y_{x,1}^m)$ the largest natural numbers such that $I_{y_{x,\bar{k}_x^m}^{(m)}} \subset f^m(I_x), I_{y_{x,-\underline{k}_x^m}^{(m)}} \subset f^m(I_x)$ and let

$$\text{Good}_x^m := \bigcup_{l=-\underline{k}_x^m}^{\bar{k}_x^m} I_{y_{x,l}^{(m)}} \quad (12)$$

$$\text{Bad}_x^m := f^m(I_x) \setminus \text{Good}_x^m \quad (13)$$

Note that

$$\frac{\nu_{f^m x}^u(\text{Bad}_x^m)}{\nu_{f^m(x)}^u(f^m(I_x))} \leq \frac{\nu_{f^m(x)}^u(\partial_{4c_{int}} f^m(I_x))}{\nu_{f^m(x)}^u(f^m(I_x))} = \frac{\nu_x^u(f^{-m}(\partial_{4c_{int}} f^m(I_x)))}{\nu_x^u(I_x)}. \quad (14)$$

Lemma 3.7. $\frac{\nu_x^u(f^{-m}(\partial_{4c_{int}} f^m(I_x)))}{\nu_x^u(I_x)} \xrightarrow{m \rightarrow \infty} 0$ uniformly in x .

Proof. Assume that this were not the case: then we could find $\zeta > 0$, a converging sequence $x_k \xrightarrow[k \mapsto \infty]{} x$ and corresponding $m_k \xrightarrow[k \mapsto \infty]{} \infty$ such that

- $\frac{\nu_{x_k}^u(f^{-m_k}(\partial_{4c_{int}} I_{f^{m_k}(x_k)}))}{\nu_{x_k}^u(I_{x_k})} \geq \zeta$.
- $\text{cl}(I_{x_k}) \xrightarrow[k \mapsto \infty]{\text{Haus}} \text{cl}(I_x)$.

For k large the set $\text{hol}_{x, x_k}^{cs}(f^{-m_k}(\partial_{4c_{int}} I_{f^{m_k}(x_k)}))$ is contained in a δ_k -neighborhood of ∂I_x inside $W^u(x)$. Since f^{-1} is uniformly contracting on unstable leaves, it follows that one can take $\delta_k \mapsto 0$, and using that the measures $\nu_{x_k}^u$ and ν_x^u are comparable we finally deduce the existence of $\zeta' > 0$ such that

$$\nu_x^u(\partial_{\delta_k} I_x) \geq \zeta'.$$

This contradicts the fact that $\nu_x^u(\partial I_x) = 0$. ■

On the other hand, by Lemma 3.5 and Eq. (11) and since the sizes of $\{\nu_x^u(I_x)\}_{x \in M}$ are uniformly comparable, we deduce the following.

Lemma 3.8. *There exists $c'_U > 0$ such that*

$$\forall x \in M, \liminf_{m \rightarrow \infty} \frac{\nu_{f^m x}^u(\{Z \in \text{Good}_x^m : Z \cap U \neq \emptyset\})}{\nu_{f^m x}^u(\{Z \in \text{Good}_x^m\})} \geq c'_U > 0.$$

Now we prove that $\Omega \neq \emptyset$.

Proposition 3.9. *There exist $\{\Theta_n^m\}_{m, n \geq 0} \in \Omega$ and $N(U) \in \mathbb{N}$, $C_U > 0$ such that for $n, m \geq N(U)$, $x \in M$ it holds*

$$\Theta_n^m(U) \geq C_U.$$

Proof. Let us start by computing

$$\begin{aligned} R_{n+m} \mathfrak{h}(x) &= \frac{1}{\nu_x^u(I_x)} \int_{I_x} \mathfrak{h} \circ f^{n+m}(t) d\nu_x^u(t) = \frac{1}{\nu_{f^m x}^u(f^m(I_x))} \int_{f^m(I_x)} \mathfrak{h} \circ f^n(t) d\nu_{f^m(x)}^u(t) \\ &= \frac{1}{\nu_{f^m(x)}^u(f^m(I_x))} \sum_{l=-\bar{k}_x^m}^{\bar{k}_x^m} \int_{I_{y_{x,l}}^{(m)}} \mathfrak{h} \circ f^n d\nu_{f^m(x)}^u + \frac{1}{\nu_{f^m(x)}^u(f^m(I_x))} \int_{\text{Bad}_x^m} \mathfrak{h} \circ f^n d\nu_{f^m(x)}^u \\ &= \frac{1}{\nu_{f^m(x)}^u(f^m(I_x))} \sum_{l=-\bar{k}_x^m}^{\bar{k}_x^m} \frac{\nu_{f^m(x)}^u(I_{y_{x,l}}^{(m)})}{\nu_{y_{x,l}}^u(I_{y_{x,l}}^{(m)})} \int_{I_{y_{x,l}}^{(m)}} \mathfrak{h} \circ f^n d\nu_{y_{x,l}}^u \\ &\quad + \frac{1}{\nu_{f^m(x)}^u(f^m(I_x))} \int_{\text{Bad}_x^m} \mathfrak{h} \circ f^n d\nu_{f^m(x)}^u. \end{aligned}$$

Denote $c_l^{(m)}(x) = \frac{\nu_{f^m(x)}^u(I_{y_{x,l}}^{(m)})}{\nu_{f^m(x)}^u(f^m(I_x))}$ and define

$$\Theta_{n,x}^m = \sum_{l=-\bar{k}_x^m}^{\bar{k}_x^m} c_l^{(m)}(x) \delta_{y_{x,l}}^{(m)} + \frac{\nu_{f^m(x)}^u(\cdot \cap \text{Bad}_x^m)}{\nu_{f^m(x)}^u(f^m(I_x))};$$

this is a probability measure supported on $f^m(I_x)$, and by the computation above

$$R_{n+m} \mathfrak{h}(x) = \int R_n \mathfrak{h}(t) d\Theta_{n,x}^m(t) + H_m^{(n)}(x)$$

with $H_m^{(n)}(x) = \frac{1}{\nu_{f^m(x)}^u(f^m(I_x))} \int_{\text{Bad}_x^m} (\mathfrak{h} \circ f^n - R_n \mathfrak{h}) d\nu_{f^m(x)}^u$, which by (14) and Proposition 3.3 satisfies

$$|H_n^{(m)}(x)| \leq 2 \|\mathfrak{h}\|_{c^0} \frac{\nu_{f^m x}^u(\text{Bad}_x^m)}{\nu_{f^m(x)}^u(f^m(I_x))} \xrightarrow{m \rightarrow \infty} 0$$

uniformly in x, n . It remains to show that $\Theta_{n,x}^m(U)$ is uniformly bounded from below: we compute

$$\Theta_{n,x}^m(U) \geq \sum_{l=-\underline{k}_x^m}^{\bar{k}_x^m} c_l^{(m)}(x)$$

which by Lemmas 3.5 and 3.8 satisfies for m large

$$\geq C_U := C_U c'_U.$$

■

We are ready to establish the SOT convergence of $\{R_n\}_n$.

Proof of Theorem 3.4. We will use the notation of the previous proposition. Letting $a_n = \inf R_n \mathfrak{h}(x)$ it follows, since $\Omega \neq \emptyset$ that $\exists \lim_n a_n = \sup a_n = a$. Proposition 3.3 and Arzelá-Ascoli's theorem imply that $\{R_n \mathfrak{h}\}_n$ is pre-compact; we will show that it only accumulates on the constant function a , thus establishing the theorem. To this end, let $(n_k)_k$ be any sub-sequence such that $\{R_{n_k} \mathfrak{h}\}_k$ is convergent to some function $\mathfrak{g} \in \mathcal{C}_c(M)$: necessarily $\min \mathfrak{g} = a$.

Observe that for every m it holds

$$R_{n_k+m} \mathfrak{h}(x) - a = \sum_{l=-\underline{k}_x^m}^{\bar{k}_x^m} c_l^{(m)}(x) \left(R_{n_k} \mathfrak{h}(y_{x,l}^{(m)}) - a \right) + H_{n_k}^{(m)}(x) - a \left(1 - \sum_{l=-\underline{k}_x^m}^{\bar{k}_x^m} c_l^{(m)}(x) \right).$$

Take x_{n_k+m} such that $R_{n_k+m} \mathfrak{h}(x_{n_k+m}) = a$: then

$$0 = \sum_l c_l^{(m)}(x_{n_k+m}) \left(R_{n_k} \mathfrak{h}(z_{k,l}^{(m)}) - a \right) + H_{n_k}^{(m)}(x_{n_k+m}) - a \left(1 - \sum_l c_l^{(m)}(x_{n_k+m}) \right). \quad (15)$$

Here the $z_{k,l}^{(m)}$ are associated to the points x_{n_k+m} . Now consider the sequence of measures

$$\Theta_{(n_k)}^m = \Theta_{n_k, x_{n_k+m}}^m,$$

Again, using that $\frac{\nu_{f^m(x)}^u(\text{Bad}_x^m)}{\nu_{f^m(x)}^u(f^m(I_x))} \xrightarrow{m \rightarrow \infty} 0$ uniformly in x , we get that $\sum_l c_l^{(m)}(x) \xrightarrow{m \rightarrow \infty} 1$ uniformly in x , and therefore

$$H_{n_k}^{(m)}(x_{n_k+m}) - a \left(1 - \sum_l c_l^{(m)}(x_{n_k+m}) \right) \rightarrow 0$$

as $m \rightarrow \infty$, uniformly for $n_k \rightarrow \infty$. By using the diagonal argument we can find sub-sequences $\{n_{k'}\}_{k'}$ of $\{n_k\}_k$, $\{m_j\} \subset \mathbb{N}$ and a measure Θ such that

$$\lim_{k' \rightarrow \infty} \lim_{j \rightarrow \infty} \Theta_{n_{k'}}^{(m_j)} = \Theta.$$

Note that by the previous proposition, $\Theta(U) \geq c'_U > 0$. Take limit as $m_j \rightarrow \infty$ and $n_k \rightarrow \infty$ simultaneously in equation (15) to get, on the one hand

$$\int (R_{n_k} \mathfrak{h} - a) d\Theta_{n_k}^{(m_j)} \rightarrow 0$$

and on the other,

$$\int (R_{n_k} \mathfrak{h} - a) d\Theta_{m_j}^{(n_k)} \rightarrow \int (\mathfrak{g} - a) d\Theta,$$

so that

$$\int (\mathfrak{g} - a) d\Theta = 0.$$

Finally, since $\min \mathfrak{g} = a$, $\mathfrak{g} - a \geq 0$ and thus

$$0 = \int (\mathfrak{g} - a) d\Theta \geq \int_U (\mathfrak{g} - a) d\Theta \geq \inf_U (\mathfrak{g} - a) \tilde{C}(U)$$

which in turn implies that $\mathfrak{g} - a$ has a zero in U . Observe that U can be taken arbitrarily small (as long as $m_\varphi(U) = 0$), therefore we deduce that $\mathfrak{g} \equiv a$ on M .

We have shown that the only accumulation point of $(R_n \mathfrak{h})_n$ is the constant function a , hence $\lim_n \|R_n \mathfrak{h} - a\|_{c^0} = 0$ as we wanted to show. ■

4 Proof of uniqueness: theorems A and B

Now we fix ξ a SLY partition for the flow, and let

$$\mathcal{N} := \bigcup_{n \geq 0} f^{-n} \bigcup_{x \in M} \partial \xi(x) \quad (16)$$

By Theorem 2.2 it holds that $m(\mathcal{N}) = 0$ for any equilibrium state m of the system, in particular for $m = m_\varphi$. For $n \in \mathbb{N}$ we consider also $\xi^n = f^n \xi$ (i.e. $x \in \mathcal{N}, \xi^n(x) = f^n \xi(f^{-n}x)$). It is also no loss of generality to assume that for every $x \notin \mathcal{N}$ the atom $\xi(x)$ contains a neighborhood of x inside $W^u(x)$.

Given $\mathfrak{h} \in \mathcal{C}(M), x \in M \setminus \mathcal{N}$ define

$$E_n \mathfrak{h}(x) = \frac{1}{\nu_x^u(\xi^n(x))} \int_{\xi^n(x)} \mathfrak{h}(t) d\nu_x^u(t) = \frac{1}{\nu_{f^{-n}x}^u(\xi(f^{-n}x))} \int_{\xi(f^{-n}x)} \mathfrak{h} \circ f^n(t) d\nu_{f^{-n}x}^u(t). \quad (17)$$

It follows that

1. $\sup_x |E_n \mathfrak{h}(x)| \leq \|\mathfrak{h}\|_{C^0}$;
2. for every $m \in \mathcal{P}r(M)$ having conditionals given by $\{\nu_x^u\}_{x \in M}$ (in particular, for every equilibrium state by Theorem 2.2), $E_n \mathfrak{h}$ is a version of the conditional expectation $\mathbb{E}_m(\mathfrak{h}|\xi^n)$.

We now argue similarly as we did in the previous part: given $x \in M \setminus \mathcal{N}$ and m large, cover $f^m(\xi(x))$ as follows: consider $I_{z_{0,x}^m}, \dots, I_{z_{o(x,m),x}^m} \subset f^m(\xi(x))$ a maximal disjoint family, and call $\text{Good}_x^m = \bigcup_{l=0}^{o(x,m)} I_{z_{l,x}^m}$, $\text{Bad}_x^m = f^m(I_x) \setminus \text{Good}_x^m$.

Observe that $\text{Bad}_x^m \subset \partial_{2c_{\text{int}}} f^m \xi(x)$, and since $\nu_x^u(\partial \xi(x)) = 0$ we can argue as in Lemma 3.7 and deduce

$$\lim_{m \rightarrow \infty} \frac{\nu_{f^m x}^u(\bigcup_{Z \in \text{Bad}_x^m} Z)}{\nu_{f^m x}^u(\bigcup_{Z \in \text{Good}_x^m} Z)} = 0. \quad (18)$$

Now compute, as in the proof Proposition 3.9,

$$\begin{aligned} E_n \mathfrak{h}(x) &= \frac{1}{\nu_x^u(\xi^n(x))} \int_{\xi^n(x)} \mathfrak{h}(t) d\nu_x^u(t) = \sum_{l=0}^{o(x,n)} \frac{\nu_x^u(I_{z_{l,x}^n})}{\nu_x^u(\xi^n(x))} R_n \mathfrak{h}(z_{l,x}^n) + \frac{1}{\nu_x^u(\xi^n(x))} \int_{\text{Bad}_x^n} \mathfrak{h}(t) d\nu_x^u(t) \\ &= \int R_n \mathfrak{h}(t) d\Theta_{n,x}(t) + T_{n,x} \end{aligned}$$

where $\Theta_{n,x}$ is a probability supported on $\xi^n(x)$ and $T_{n,x}$ converges to zero as $n \mapsto \infty$, due to Eq. (18). Taking an accumulation point of $\{\Theta_{n,x}\}$ and using the uniform convergence of $\{R_n \mathfrak{h}\}_{n \geq 0}$ we deduce

$$\lim_n E_n \mathfrak{h}(x) = a \quad \forall x \notin \mathcal{N}. \quad (19)$$

Proof of Theorem B. Let $m \in \mathcal{P}r(M)$ be such that its conditionals in ξ are given by $\{\nu_x^u\}_x$, i.e.

$$m_x^\xi = \nu_x^u(\cdot | \xi(x)) \quad m\text{-a.e.}(x).$$

In this case, $m^{\xi^n} = \nu_x^u(\cdot | \xi^n(x))$. Now for $\mathfrak{h} \in \mathcal{C}(M)$ we can compute

$$\int \mathfrak{h} dm = \int E_n \mathfrak{h}(x) dm(x) \xrightarrow{n \rightarrow \infty} \int a dm(x) = a$$

where we have used (19) and the fact $m(M \setminus \mathcal{N}) = 1$. Since a does not depend on m we conclude

$$\int \mathfrak{h} dm = \int \mathfrak{h} dm_\varphi \quad \forall \mathfrak{h} \in \mathcal{C}(M)$$

and therefore, $m = m_\varphi$. ■

Proof of Theorem A. Let $\tilde{\mu}^{cs} = \{\tilde{\mu}_x^{cs}\}_{x \in M}$ be a quasi-invariant measure for Φ^u with Jacobian determined by $h \in \mathcal{C}^0\text{-}c(\Phi)$. Construct the corresponding probability measure m on M as explained in the last part of the Second section. By construction m has conditionals given by the family $\{\nu_x^u\}$, hence by Theorem B $m = m_{\Phi, \varphi}$. From here we deduce, using the local product structure of m that $\tilde{\mu}^{cs} = \mu^{cs}$. ■

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