Rigidity of equilibrium states and unique quasi-ergodicity for horocyclic foliations

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April 26, 2023

Abstract

In this paper we prove that for topologically mixing Anosov flows their equillibrium states corresponding to Hölder potentials satisfy a strong rigidity property: they are determined only by their disintegrations on (strong) stable or unstable leaves.

As a consequence we deduce: the corresponding horocyclic foliations of such systems are uniquely quasi-ergodic, provided that the corresponding Jacobian is Hölder, without any restriction on the dimension of the invariant distributions. This generalizes a classical result due to Babillott and Ledrappier for the geodesic flow of hyperbolic manifolds.

We rely on symbolic dynamics and on recent methods developed by the authors.

Let $\mathtt{f} = (f_t)_t : M \to M$ be an Anosov flow with invariant bundle decomposition $TM = E^s \oplus E^c \oplus E^u$. As it is well known, both E^s, E^u are integrable to leafwise smooth foliations $\mathcal{W}^s, \mathcal{W}^u$ which in this work we refer as the $s-, u-$ horocyclic foliations. We assume that $\mathtt{f}$ is topologically mixing: this is equivalent to the fact that every leaf of its u-horocyclic (or s-horocyclic) foliation is dense in M (minimality of the corresponding foliation).

To keep the presentation short it is expected that the reader is acquainted with the basic theory of thermodynamic formalism, particularly for hyperbolic systems as covered in [3], and the theory of measurable partitions in the sense of Rokhlin [11]. We remind the reader that given an invariant measure for $\mathtt{f}$ one can always find (increasing) measurable partitions whose atoms are subsets of the leaves of $\mathcal{W}^u$ and contain relatively open neighborhoods of each point, for almost every point. See for example the discussion in [6]. We say that such partitions are adapted to $\mathcal{W}^u$.

In this work we prove the following theorem related to the thermodynamic formalism of hyperbolic systems.

Main Theorem. *Assume $\mathtt{f} = (f_t)_t : M \to M$ to be a $\mathcal{C}^2$ Anosov flow that is topologically mixing, and let $\varphi : M \to \mathbb{R}$ be a Hölder function. Then there exists a family $\{\nu^u_x\}_{x\in M}$ satisfying:*

1. *ν^u_x is a Radon measure on the $W^u(x)$.*
2. *If m is any probability, invariant or not, and ξ is a measurable partition adapted to $\mathcal{W}^u$ whose conditionals are of the form*
$$\mathrm{m}^\xi_x = \frac{\nu^u_x}{\nu^u_x(\xi(x))},$$
then m is the unique equilibrium state for the system $(\mathtt{f}, \varphi)$.

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This result a strong formulation of the classical Sinai-Ruelle-Bowen theorem, since it does not require invariance of the measure. In [5] it was obtained under the assumption $\dim E^u = 1$; here we remove this hypothesis. However, the methods in the aforementioned article are geometrical, while in the present article we rely on the theory of symbolic dynamics. This powerful tool is not available (at least at its peak) for more general systems, such as partially hyperbolic maps, which was the main reason for the authors to avoid using it before. Still, due to the importance of thermodynamic formalism for classical hyperbolic systems, it is valuable to have the previous theorem without restrictions in the dimension of the unstable bundle.

In the same article it was also noted an interesting consequence of the above theorem for the dynamics of the foliation $\mathcal{W}^u$, which generalizes the unique ergodicity of horocyclic flows corresponding to algebraic geodesic flows discovered by Furstenberg [7], or more generally, the unique ergodicity of the strong unstable foliation for hyperbolic systems due to Bowen and Marcus [4]. Let us explain this.

All foliations considered have continuous tangent bundle, and therefore the notion of an embedded disc $D \subset M$ to be transverse to a foliation is just transversality (in the differential geometrical sense) of D to every leaf that it intersects. If $\mathcal{F}$ is a foliation we say that a family $\{\mu_x : x \in M\}$ is a *transverse measure* if it satisfies:

1. each $x \in M$ is the center of some transverse disc D_x, and
2. μ_x is a non-trivial Borel measure on D_x.

Given one of such measures and $y \in \mathcal{F}(x)$, one can use the holonomy of the foliation to compare μ_x and μ_y, and least for nearby x, y: we denote $\mathrm{hol}^{\mathcal{F}}_{y,x} : E_x \subset D_x \to E_y \subset D_y$, the holonomy transport, where E_x, E_y are relatively open.

Definition. *The transverse measure $\{\mu_x\}_x$ of $\mathcal{F}$ is quasi-invariant if there exist a family of positive functions $\mathrm{Jac}_\mu = \{\mathrm{Jac}_{y,x} : E_x \to \mathbb{R}_{>0} : y \in \mathcal{F}(x)\}$ and positive constants $\{C(y,x) : y \in \mathcal{F}(x)\}$ such that for $y \in \mathcal{F}(x)$,*

$$\mathrm{hol}^{\mathcal{F}}_{x,y}\mu_y = C(y,x)\mathrm{Jac}_{x,y}\mu_x.$$

The family Jac_ν is the Jacobian of the quasi-invariant measure. If $\mathrm{Jac}_\nu \equiv 1$ then the measure is said to be an invariant transverse measure.

In the setting that we are working it is convenient to use center stable discs as the family of transversals to $\mathcal{W}^u$, and we use implicitly this choice in what follows. As was noted in our previous article, the natural family of Jacobians for $\mathcal{W}^u$ can be constructed as we explain now. Let

$$\mathscr{C}\!\mathscr{o}\mathscr{c}(\mathtt{f}) = \{h : M \to \mathbb{R}_+ : h(x) = e^{\int_0^1 \varphi(f_t x)dt} \text{ for some Hölder function } \varphi\}, \tag{1}$$

and given $h \in \mathscr{C}\!\mathscr{o}\mathscr{c}(\mathtt{f})$, define

$$\hbar = \{H_{x_0,y_0} : x_0, y_0 \in M, y_0 \in W^u(x_0)\} \tag{2}$$

with

$$H_{x_0,y_0}(x) = \prod_{j=1}^{\infty} \frac{h(f_{-j} \circ \mathrm{hol}^{\mathcal{W}^u}_{y_0,x_0} x)}{h(f_{-j}(x))}, \quad x \in W^{cs}(x_0), \mathrm{hol}^{\mathcal{W}^u}_{y_0,x_0} x \in W^{cs}(y_0). \tag{3}$$

Main Corollary. *In the hypotheses of the Main Theorem, given $h \in \mathscr{C}\!\mathscr{o}\mathscr{c}(\mathtt{f})$ there exists $\mu^{cs} = \{\mu^{cs}_x\}$ a transverse measure for $\mathcal{W}^u$ such that μ^{cs} is the unique quasi-invariant measure with Jacobian given by the family $\hbar$ determined by h.*

The proof that the Main Theorem implies the Main Corollary is exactly the same as the one given in [5], and thus we will not repeat it here. We remark that this unique-quasi ergodicity was known for unstable foliations corresponding to Abelian covers of the geodesic flow hyperbolic closed manifolds cf. [1, 13], where it is used strongly the symmetries coming the geometry of such systems. The result for general Anosov flows remained open until now. We also point out that a complete discussion of the significance of this theorem, as well as a comparison with other literature can be found in the recent series of articles of the authors cited above, hence we refer the reader to these for more background.

We finish this introduction by posing a general question. It follows by the results of this article that horocyclic foliations have strong rigidity properties in terms of their admitted quasi-invariant measures. On the other hand, the existence of the renormalizing dynamics (i.e., the associated hyperbolic system) is probably not necessary for this phenomena, as the classical result of M. Ratner [10] suggest. We can thus ask:

Question. *Let Γ be a co-compact lattice in a lie group G, and let U be a one-parameter unipotent subgroup of G acting minimally on Γ/G. Consider α a differentiable multiplicative cocycle over $U \curvearrowright \Gamma/G$. Does there exists a unique quasi-invariant measure for the action with Jacobian given by α?.*

Ratner answers positively the previous question when the cocycle is trivial.

1 Prerequisites

Given a matrix $A \in \mathtt{Mat}_d(\{0,1\})$ one considers the two-sided subshift of finite type that it determines,

$$\Sigma_A = \{\underline{x} = (x_n)_n \in \{0,1\}^{\mathbb{Z}} : A_{x_n x_{n+1}} = 1, \forall n \in \mathbb{Z}\}.$$

For $k, l \in \mathbb{N}$ and a word $a_{-k} \cdot a_{-k+1} \cdots a_l \in \{0,1\}^{l+k+1}$ with $A_{a_i, a_{i+1}} = 1$ for all i,we denote

$$C(a_{-k} \cdots a_l)_{-k} = \{\underline{x} \in \Sigma_A : x_i = a_i, \forall -k \leq i \leq l\}.$$

Note that

$$C(a_{-k} \cdots a_l)_{-k} = \bigcap_{i=-k}^{l} \sigma^{-i}(C(a_i)_0).$$

Definition 1.1. *Sets of the previous form are called rectangles. If $k = l$ we say that the rectangle is symmetric.*

The topology in Σ_A is metrizable, where a compatible metric is given as

$$d(\underline{x}, \underline{y}) = \frac{1}{2^{N+1}};$$

above N is the size of the largest symmetric rectangle that contains $\underline{x}, \underline{y}$.

Similar considerations can be applied to the one-sided shift spaces

$$\Sigma_A^- = \{\underline{x} = (x_n)_{n \leq 0} \in \{0,1\}^{-\mathbb{N}} : A_{x_n x_{n+1}} = 1, \forall n < 0\}$$
$$\Sigma_A^+ = \{\underline{x} = (x_n)_{n > 0} \in \{0,1\}^{\mathbb{N}^*} : A_{x_n x_{n+1}} = 1, \forall n > 0\}$$

Note that any $\underline{x} \in \Sigma_A$ can be written uniquely in the form

$$\underline{x} = \underline{x}^- \cdot \underline{x}^+ \quad \underline{x}^- \in \Sigma_A^-, \underline{x}^+ \in \Sigma_A^+$$

where the $\cdot$ denotes concatenation.

In both Σ_A, Σ_A^+ there is a continuous homeomorphism, respectively a d-to-1 endomorphism (called simply the shift) given as

$$\sigma(\underline{x}) = (x_{n+1})_n,$$

whereas in Σ_A^- we use the inverse σ^{-1}.

Definition 1.2. *A is mixing*[1] *if there exists $M \in \mathbb{N}$ such that every entry of A^M is positive.*

It is easy to verify that A is mixing if and only if $\sigma : \Sigma_A \to \Sigma_A$ is topologically mixing.

For $\underline{x} \in \Sigma_A$ we denote its local stable/unstable sets by

$$W^s_{\text{loc}}(\underline{x}) = \{\underline{y} \in \Sigma_A : x_n = y_n, \forall n \geq 0\}$$
$$W^u_{\text{loc}}(\underline{x}) = \{\underline{y} \in \Sigma_A : x_n = y_n, \forall n \leq 0\}.$$

A central remark in the theory is that one can consider Σ_A^+ as the space of local unstable sets of Σ_A: for $\underline{y} \in W^u_{\text{loc}}(\underline{x})$, $\underline{y} = \underline{x}^- \cdot \underline{y}^+$.

Given $\phi : \Sigma_A \to \mathbb{R}$ and $n \geq 0$ let

$$\text{var}_n(\phi) = \sup\{|\phi(\underline{x}) - \phi(\underline{y})| : x_k = y_k, \forall |k| \leq n\},$$

and denote

$$\mathcal{F}_A = \{\phi : \text{var}_n(\phi) \leq C\theta^n, \text{ for some } C, \theta > 0\}.$$

Similarly we can define the function spacees $\mathcal{F}_A^*, * \in \{-, +\}$. We recall the following classical lemma (see for example lemma 1.6 in [3]).

Lemma 1.1. *If $\phi \in \mathcal{F}_A$ then there exists functions $\phi^*, \gamma^* : \Sigma_A \to \mathbb{R}, * \in \{-, +\}$ such that $\phi^* - \phi = \gamma^* - \gamma^* \circ \sigma$, and*

- *ϕ^+ depends only on the coordinates > 0,*
- *ϕ^- depends only on the coordinates ≤ 0.*
- *$\phi^* \in \mathcal{F}_A^*$.*

It follows that we can identify ϕ^* as an element in $\mathcal{F}_A^*$.

1.1 Suspension

If $R : \Sigma_A \to \mathbb{R}_{>0}$ is continuous, we consider the space

$$\{(\underline{x}, t) \in \Sigma_A \times \mathbb{R} : 0 \leq t \leq R(\underline{x})\}$$

and identify

$$(\underline{x}, R(\underline{x})) \sim (\sigma \underline{x}, 0)$$

to obtain a compact bundle over the circle, $S(A, R) = \Sigma_A \to \mathbb{R}_{>0}/\sim$, together with the natural flow

$$s_t([\underline{x}, u]) = [\underline{x}, u + t].$$

Definition 1.3. *The space $S(A, R)$ is the suspension of the shift map $\sigma : \Sigma_A \to \Sigma_A$ under the roof function R. The flow s_t is the suspension flow.*

Remark 1.1. *The sets*

$$D(i) = \{[\underline{x}, 0] : x_0 = i\}, \quad 1 \leq i \leq d$$

are pairwise disjoint, and $T = \bigcup_{i=1}^d D(i)$ are global transversal to the flow s_t, in the sense that every orbit of s_t intersects T.

[1] Equivalently, it is irreducible and aperiodic.

Write, using Lemma 1.1,
$$R^* - R = v^* - v^* \circ \sigma \quad * \in \{+,-\}$$
and construct the spaces

$$S^*(A,R) = \{(\underline{x},t) \in \Sigma_A \times \mathbb{R} : 0 \leq t \leq R^*(\underline{x})\}/\sim$$

by identifying $(\sigma^{-1}\underline{x}, R^-(\underline{x})) \sim (\underline{x}, 0)$ in $S^-(A,R)$, and $(\sigma(\underline{x}),0) \sim (\underline{x}, R^+(\underline{x}))$ in $S^+(A,R)$. Similarly as in the case of $S(A,R)$ one obtains semi-flows in $S^-(A,R), S^+(A,R)$.

We can characterize easily the local unstable manifolds of a point in the suspension, when $R \in \mathcal{F}_A$. Define the map $\Psi : \Delta \subset S^-(A,R) \times \Sigma_A^+ \to S(A,R)$ by

$$\Psi([\underline{x},t],\underline{y}) = [\underline{z}, t + v^-(\underline{z})],$$

where $\underline{z} = \underline{x} \cdot \underline{y} \in \Sigma_A$ (assuming that $A_{x_0,y_1} = 1$). Then Ψ is an homeomorphism and sends Σ_A^+ to the local unstable foliation in $S(A,R)$.

We finish this part recalling the following important theorem.

Theorem 1.2 ([2, 9]). *Given an Anosov flow $\mathtt{f} = (f_t)_t : M \to M$ and $\epsilon > 0$, there exists $A \in \mathtt{Mat}_d(\{0,1\})$, $R \in \mathcal{F}_A$ strictly positive and a Hölder continuous function $\pi : S(A,R) \to M$ such that*

- *π is surjective, uniformly bounded to one, and $1-1$ on a residual set.*
- *$\pi \circ s_t = f_t \circ \pi, \forall t.$*
- *$\operatorname{diam} \pi(D(i)) < \epsilon, \forall 1 \leq i \leq d$*

If $\mathtt{f}$ is topologically mixing then A is mixing.

Given an Anosov flow $\mathtt{f}$ with symbolic model $\pi : S(A,R) \to M$ as in the previous theorem, the i-th rectangle is
$$\mathtt{R}_i = \pi(D(i)),$$
and it follows that $\tilde{T} = \bigcup_{i=1}^d \mathtt{R}_i$ is a global transversal to $\mathtt{f}$; one can even make the construction so that the relative interior of each R_i is smooth.

1.2 Ruelle-Perron-Frobenius operator

For a continuous map of a compact metric space $f : M \to M$ and a continuous function $\varphi : M \to \mathbb{R}$ we denote

1. $\mathscr{Pr}_f(M)$ the set of f-invariant probability measures,
2. $P_{top}(\varphi) = \sup_{\nu \in \mathscr{Pr}_f(M)}\{h_\nu(f) + \int \varphi d\nu\}$ the topological pressure of the system (f,φ),
3. $\mathscr{Eq}(f,\varphi) = \{\mu \in \mathscr{Pr}_f(M) : P_{top}(\varphi) = h_\mu(f) + \int \varphi d\mu\}$, the set of equilibrium states of (f,φ).

Correspondingly, for a flow $\mathtt{f} = (f_t)_t$ on M we write

4. $\mathscr{Pr}_{\mathtt{f}}(M) = \bigcap_t \mathscr{Pr}_{f_t}(M)$, the set of flow invariant measures,
5. $P_{top}(\mathtt{f},\varphi) = \sup_{\nu \in \mathscr{Pr}_{\mathtt{f}}(M)}\{h_\nu(f_1) + \int \varphi d\nu\}$, the topological pressure of the system $(\mathtt{f},\varphi)$,
6. $\mathscr{Eq}(\mathtt{f},\varphi)$ the set of equilibrium states of the system $(\mathtt{f},\varphi)$.

Theorem 1.3 (Proposition 6.2 in [6]). *If* $\mathtt{f} = (f_t)_t$ *is a topologically mixing Anosov flow and* φ *is Hölder, then* $\mathscr{E}q(\mathtt{f}, \varphi) = \mathscr{E}q(f_1, \varphi^1) = \{\mathrm{m}_\varphi\}$*, where*

$$\varphi^1(x) = \int_0^1 \varphi(f_t x)\,\mathrm{d}t.$$

From now on $\mathtt{f} = (f_t)_t$ denotes an Anosov flow in the hypotheses of the Main Theorem with symbolic model $\pi : S(A, R) \to M$, and $\varphi : M \to \mathbb{R}$ is a fixed Hölder function. It is not loss of generality (by subtracting the pressure to φ) to assume

$$P_{top}(\mathtt{f}, \varphi) = 0.$$

We recall one possible construction of the equilibrium state associated to the system $(\mathtt{f}, \varphi)$. The map $\phi = \varphi \circ \pi$ is Hölder continuous, and one is led to consider equilibrium states for suspended flows. Due to the structure of these, it is enough to consider equilibrium states for $\sigma : \Sigma_A \to \Sigma_A$, replacing the potential ϕ by its integrated version

$$\tilde{\phi}(\underline{x}) = \int_0^{R(\underline{x})} \phi([\underline{x}, u])du.$$

Using the (local) product structure of $S(A, R)$, one defines also $\psi : \Delta' \subset \Sigma_A^- \times S^+(A, R) \to \mathbb{R}$ with

$$\psi(\underline{x}, [\underline{y}, t]) = \phi(\underline{x} \cdot \underline{y}, t - v^+(\underline{x} \cdot \underline{y}))$$

together with its integrated version

$$\tilde{\psi}(\underline{x} \cdot \underline{y}) = \int_0^{R^+(\underline{y})} \psi(\underline{x}, [\underline{y}, u])du.$$

It turns out that $\tilde{\psi} - \tilde{\phi} = k - k \circ \sigma$, with

$$k(\underline{x} \cdot \underline{y}) = \int_{-v^+(\underline{x}\underline{y})}^{0} \phi([\underline{x} \cdot \underline{y}, u])du.$$

If follows then that $\mathscr{E}q(\sigma, \tilde{\phi}) = \mathscr{E}q(\sigma, \tilde{\psi})$.

We can now apply the technology of transfer operators: write $\tilde{\psi}^+ = \tilde{\psi} + w - w \circ \sigma$, and for continuous valued functions $h \in \mathcal{C}(\Sigma_A^+)$ let

$$\mathcal{L}h(\underline{x}) = \sum_{\sigma \underline{y} = \underline{x}} e^{\tilde{\psi}^+(\underline{y})} h(\underline{y}).$$

Then $\mathcal{L}\mathbb{1} = \mathbb{1}$, and the equilibrium state μ for the system $(\Sigma_A^+, \tilde{\psi}^+)$ is the unique eigen-measure of the adjoint of $\mathcal{L}$. The corresponding equilibrium state for the system $(\sigma, \tilde{\phi})$ is then the (unique) shift invariant extension of μ to Σ_A, which we also denote as μ.

The approach of N. Haydn (see [8]) is then defining the measure $\mu^u_{\Psi([\underline{x},0],\underline{y})}$ on $W^u_{\mathrm{loc}}(\Psi([\underline{x}, 0], \underline{y}))$ as

$$\Psi_* e^{-(w+v)} \mu$$

and for a general point $\Psi([\underline{x}, t], \underline{y})$ with $0 \leq t \leq R^+(\underline{x})$ we define $\mu^u_{\Psi([\underline{x},t],\underline{y})}$, so that

$$(s_{-t})_* \mu^u_{\Psi([\underline{x},t],\underline{y})} = e^{-\int_0^t \phi([\cdot,u])\,\mathrm{d}u} \mu^u_{\Psi([\underline{x},0],\underline{y})}$$

Remark 1.2. *The function* $w + v$ *is the transfer function between the cohomologous potentials* $\tilde{\phi}, \tilde{\psi}^+$.

Theorem 1.4. *The family of measures $\mu^u = \{\mu^u_p : p \in S(A,R)\}$ satisfies:*

1. *each μ^u_p is non-atomic with full support in $W^u_{\mathrm{loc}}(p)$.*
2. *It holds*
$$(s_{-t})_*\mu^u_{s_t(p)} = e^{-\int_0^t \phi([\cdot,u])du}\mu^u_p.$$
3. *The family μ^u depends continuously on p*

We need to modify the previous family; denote

1. $h_\phi(\underline{z}) = e^{\tilde{\phi}(z)}$,
2. $\Delta_{\underline{x},\underline{y}} : W^u_{\mathrm{loc}}(\Psi([\underline{x},0],\underline{y})) \to \mathbb{R}_{>0}$,
$$\Delta_{\underline{x},\underline{y}}(\Psi([\underline{x},0],\underline{y}')) := \prod_{k=1}^{\infty} \frac{h_\phi(\sigma^{-k}\underline{x}\cdot y')}{h_\phi(\sigma^{-k}\underline{x}\cdot y)}.$$
3. $\nu^u_{\Psi([\underline{x},0],\underline{y})} = \Delta_{\underline{x},\underline{y}}\mu^u_{\Psi([\underline{x},0],\underline{y})}$

and extend for points $p = \Psi([\underline{x},t],\underline{y}), 0 \le t < R^+(\underline{x})$ by requiring
$$(s_{-t})_*\nu^u_{s_t(p)} = e^{-\int_0^t \phi([\underline{x}\cdot\underline{y},u])du}\nu^u_p$$
By the form of these measures, there is a one to one correspondence with a family $\{\nu^u_{\underline{z}} : \underline{z} \in \Sigma_A\}$ satisfying the following properties.

Proposition 1.5. *It holds:*

1. *each $\nu^u_{\underline{z}}$ is a measure on $W^u_{\mathrm{loc}}(\underline{z})$.*
2. $\sigma\nu^u_{\underline{z}} = e^{\tilde{\phi}(z)}\nu^u_{\sigma\underline{z}}$.
3. *If $\xi = \{W^u_{\mathrm{loc}}(\underline{z}) : \underline{z} \in \Sigma_A\}$ and $\mathscr{Eq}(\sigma,\phi) = \{\mathrm{m}\}$, then the conditional measures of m on ξ are given as*
$$\mathrm{m}^\xi_{\underline{z}} = \frac{\nu^u_{\underline{z}}}{\nu^u_{\underline{z}}(W^u_{\mathrm{loc}}(\underline{z}))},$$

Proof. The first and the second part follow directly by construction. The third is consequence of the previous ones, and is proven (in more generality) in Section 4 of [6]. ■

Using π we can use the measures ν^u_p to induce a family $\{\nu^u_x : x \in M\}$ where each ν^u_x is a measure on a local unstable disc containing $x \in M$. This follows from the properties of the symbolic model, and is given for example in the Appendix of [12].

The measures defined in the Main Theorem are precisely the ν^u_x. Note that these give the disintegration of the unique equilibrium state for the system $(\mathtt{f},\varphi)$, and therefore they are uniquely defined modulo normalization.

Remark 1.3. *Suppose that $\mathrm{m} \in \mathscr{Pr}(M)$ for which there exists an adapted partition ξ to $\mathcal{W}^u$ so that its conditionals are of the form*
$$\mathrm{m}^\xi_x = \frac{\nu^u_x}{\nu^u_x(\xi(x))}.$$
Then the projetive class of ν^u_x can be recovered from the conditional measures, and in particular if ξ' is another adapted partition to $\mathcal{W}^u$, then the conditional measures of m with respect to ξ' are given by $\{\nu^u_x\}$.

From the previous discussion it follows that to establish the Main Theorem it suffices to prove:

Theorem 1.6. *Let* $\mathrm{m} \in \mathscr{P}r(\Sigma_A)$ *so that its disintegration along the partition* ξ *is given by the family* $\nu^u_{\underline{z}}$*,*

$$\mathrm{m}^\xi_{\underline{z}} = \frac{\nu^u_{\underline{z}}(\cdot)}{\nu^u_{\underline{z}}(W^u_{\mathrm{loc}}(\underline{z}))}.$$

Then m *is the unique equilibrium state for the system* (σ, ϕ).

We finish this part by noting that since $\sigma W^u_{\mathrm{loc}}(\underline{x}) = \bigsqcup_{i=1}^d W^u_{\mathrm{loc}}(\underline{x}^i)$ for some points $\underline{z}^i$, we can extend the $\nu^u_{\underline{z}}$ to measures on the whole unstable set $W^u(\underline{z})$. We will tacitly assume that in what follows.

2 Marcus' operators

Given a continous function $\mathfrak{h} : \Sigma_A \to \mathbb{R}$ and $n \geq 0$ we define a new continous function

$$R_n\mathfrak{h}(\underline{x}) = \frac{1}{\nu^u_{\underline{x}}(W^u_{\mathrm{loc}}(\underline{x}))} \int_{W^u_{\mathrm{loc}}(\underline{x})} \mathfrak{h} \circ \sigma^n \,\mathrm{d}\nu^u_{\underline{x}}. \tag{4}$$

This way we get a family of linear contractions $\{R_n : \mathcal{C}(\Sigma_A) \to \mathcal{C}(\Sigma_A) : n \geq 0\}$.

Proposition 2.1. *The family* $\{R_n\mathfrak{h}\}_n$ *is equicontinuous.*

Proof. Let $\underline{x}, \underline{y}, \underline{z} \in C(a)_0$ with

- $\underline{y} \in W^u_{\mathrm{loc}}(\underline{x})$,
- $\underline{z} \in W^u_{\mathrm{loc}}(\underline{y})$

We have $\nu^u_{\underline{x}} = C(\underline{x}, \underline{y})\nu^u_{\underline{y}}$, and thus $R_n\mathfrak{h}(\underline{x}) = R_n\mathfrak{h}(\underline{y})$. On the other hand, the difference between the functions $\mathfrak{h} \cdot \mathbb{1}_{W^u_{\mathrm{loc}}(\underline{y})}, \mathfrak{h} \cdot \mathbb{1}_{W^u_{\mathrm{loc}}(\underline{z})}$ converges uniformly to zero as $\underline{y} \mapsto \underline{z}$, and it follows that same is true for the quantity

$$\left| \frac{1}{\nu^u_{\underline{y}}(W^u_{\mathrm{loc}}(\underline{y}))} \int_{W^u_{\mathrm{loc}}(\underline{y})} \mathfrak{h} \,\mathrm{d}\nu^u_{\underline{y}} - \frac{1}{\nu^u_{\underline{z}}(W^u_{\mathrm{loc}}(\underline{z}))} \int_{W^u_{\mathrm{loc}}(\underline{z})} \mathfrak{h} \,\mathrm{d}\nu^u_{\underline{z}} \right|$$

For points $\underline{y}, \underline{z}$ in the same stable set one can apply this fact to $\mathfrak{h} \circ \sigma^n \cdot \mathbb{1}_{W^u_{\mathrm{loc}}(\underline{y})}, \mathfrak{h} \circ \sigma^n \mathbb{1}_{W^u_{\mathrm{loc}}(\underline{z})}, \forall n \geq 0$.

Putting everything together and using the local product structure inside $C(a)_0$ we deduce the claim. ■

We will now show that $\{R_n\mathfrak{h}\}_n$ converges uniformly to some constant $c(\mathfrak{h})$. Let Ω be the set of all sequences $\{\Theta^m_{n,\underline{x}} : n, m \geq 0, \underline{x} \in \Sigma_A\}$ satisfying

1. $\Theta^m_{n,x}$ is a probability measure on $\sigma^m(W^u_{\mathrm{loc}}(\underline{x}))$,
2. for every $\underline{x} \in \Sigma_A$

$$R_{n+m}\mathfrak{h}(\underline{x}) = \int R_n\mathfrak{h}(\underline{y}) \,\mathrm{d}\Theta^m_{n,\underline{x}}(\underline{y}). \tag{5}$$

Definition 2.1. $\{\Theta^m_{n,\underline{x}}\} \in \Omega$ *is adapted to a cylinder* U *if*

$$\inf_{n,m,\underline{x}} \Theta^m_{n,\underline{x}}(U) > 0.$$

The bulk of the work is contained in proving that for any cylinder U there is $\{\Theta^m_{n,\underline{x}}\} \in \Omega$ adapted to it. If this were the case, note that $\{c_n(\mathfrak{h}) = \inf_{\underline{x}} R_n\mathfrak{h}(\underline{x})\}_{n\geq 0}$ is an increasing bounded sequence of real numbers, and therefore

$$\exists c(\underline{h}) = \lim_n c_n(\mathfrak{h}) = \sup_n c_n(\mathfrak{h}).$$

Note also that if g is any accumulation point of $\{R_n\mathfrak{h}\}_n$, then $g \geq c(\underline{h})$.

Recall that A^m has positive entries, for every $m \geq M$.

Lemma 2.2. *Given a cylinder U there exist $C_U > 0$ such that for every $\underline{x} \in \Sigma_A, \forall m > M$ it holds*

$$\sigma^m(W^u_{\text{loc}}(\underline{x})) = \bigcup_{i=1}^{k_m} W^u_{\text{loc}}(\underline{y}^i)$$

with

$$\frac{\#\{i : \underline{y}^i \in U\}}{k_m} \geq C_U.$$

Proof. It is no loss of generality to consider the case $U = C(a_0 \cdots a_l)_0$, since $\sigma^k C(a_0 \cdots a_l)_{-k} = C(a_0 \cdots a_l)_0$. Fix $\underline{x}$: we have

$$\sigma^m(W^u_{\text{loc}}(\underline{x})) = \{\underline{y} : y_{n-m} = x_n, \forall n \leq 0\} = \bigcup_{i=1}^{k_m} W^u_{\text{loc}}(\underline{z}^i) = \bigcup_{i=1}^{k_m} W^u_{\text{loc}}(\underline{z}^i)$$

where in particular for $i \neq i'$ there exists $-m < j \leq 0$ so that $z^i_j \neq z^{i'}_j$. We denote $\mathtt{W} = \{\underline{z}^i : 1 \leq i \leq k_m\}$ and observe that we can write

$$\mathtt{W} = \bigcup_{e=1}^{d} \mathtt{W}_e \quad \mathtt{W}_e = \{\underline{z}^i \in \mathtt{W} : z^i_{-M} = e\},$$

where each $\mathtt{W}_e \neq \emptyset$. Therefore for $m \geq M$ and each $1 \leq e \leq d$ the proportion

$$\frac{\#\{\underline{z}^i \in \mathtt{W}_e : z^i_0 = a_0\}}{\#\mathtt{W}_e}$$

is positive and independent of m. This implies the claim, since one can choose a fixed proportion of points $\underline{y}^i \in W^u_{\text{loc}}(\underline{z}^i)$ from those with $z^i_0 = a_0$, also satisfying $\underline{y}^i \in U$ (because U does not depend on m). ■

We compute, for $n, m \geq 0$

$$R_{n+m}\mathfrak{h}(\underline{x}) = \frac{1}{\nu^u_{\underline{x}}(W^u_{\text{loc}}(\underline{x}))} \int_{W^u_{\text{loc}}(\underline{x})} \mathfrak{h} \circ \sigma^{n+m} \, \mathrm{d}\nu^u_{\underline{x}} = \frac{1}{\nu^u_{\sigma^m\underline{x}}(\sigma^m(W^u_{\text{loc}}(\underline{x})))} \int_{\sigma^m(W^u_{\text{loc}}(\underline{x}))} \mathfrak{h} \circ \sigma^n \, \mathrm{d}\nu^u_{\sigma^m\underline{x}}$$

$$= \frac{1}{\nu^u_{\sigma^m\underline{x}}(\sigma^m(W^u_{\text{loc}}(\underline{x})))} \sum_{i=1}^{k_m} \int_{W^u_{\text{loc}}(\underline{y}^i)} \mathfrak{h} \circ \sigma^n \, \mathrm{d}\nu^u_{\sigma^m\underline{x}}$$

$$= \frac{1}{\nu^u_{\sigma^m\underline{x}}(\sigma^m(W^u_{\text{loc}}(\underline{x})))} \sum_{i=1}^{k_m} \frac{\nu^u_{\sigma^m\underline{x}}(W^u_{\text{loc}}(\underline{y}^i))}{\nu^u_{\underline{y}^i}(W^u_{\text{loc}}(\underline{y}^i))} \int_{W^u_{\text{loc}}(\underline{y}^i)} \mathfrak{h} \circ \sigma^n \, \mathrm{d}\nu^u_{\underline{y}^i}$$

$$= \sum_{i=1}^{k_m} \frac{\nu^u_{\sigma^m\underline{x}}(W^u_{\text{loc}}(\underline{y}^i))}{\nu^u_{\sigma^m\underline{x}}(\sigma^m(W^u_{\text{loc}}(\underline{x})))} R_n\mathfrak{h}(\underline{y}^i).$$

It follows that $\{\Theta^m_{n,\underline{x}}\} \in \Omega$, where

$$\Theta^m_{n,\underline{x}} = \sum_{i=1}^{k_m} \frac{\nu^u_{\sigma^m\underline{x}}(W^u_{\text{loc}}(\underline{y}^i))}{\nu^u_{\sigma^m\underline{x}}(\sigma^m(W^u_{\text{loc}}(\underline{x})))} \delta_{\underline{y}^i}.$$

Lemma 2.3. *$\{\Theta^m_{n,\underline{x}}\}$ is adapted to U*

Proof. Indeed, by definition of the measure $\nu^u_{\sigma^m \underline{x}}$,

$$\Theta^m_{n,\underline{x}}(U) = \frac{\sum_{\substack{i=1 \\ \underline{y}^i \in U}}^{k_m} \nu^u_{\sigma^m \underline{x}}(W^u_{\mathrm{loc}}(\underline{y}^i))}{\sum_{i=1}^{k_m} \nu^u_{\sigma^m \underline{x}}(W^u_{\mathrm{loc}}(\underline{y}^i))} = \frac{\sum_{\substack{i=1 \\ \underline{y}^i \in U}}^{k_m} \nu^u_{\underline{y}^i}(W^u_{\mathrm{loc}}(\underline{y}^i))}{\sum_{i=1}^{k_m} \nu^u_{\underline{y}^i}(W^u_{\mathrm{loc}}(\underline{y}^i))}.$$

On the other hand, for every $\underline{z}, \underline{w} \in \Sigma_A$ the measures $\nu^u_{\underline{z}}(W^u_{\mathrm{loc}}(\underline{z})), \nu^u_{\underline{w}}(W^u_{\mathrm{loc}}(\underline{w}))$ are uniformly comparable, therefore the result follows from Lemma 2.2. ■

Consider any point of accumulation $\mathfrak{g} \in \mathcal{C}(\Sigma_A)$ of $\{R_n \mathfrak{h}\}$, with

$$\lim_k R_{n_k} \mathfrak{h} = \mathfrak{g}.$$

As noted $g \geq c(\underline{h})$, and one can compute

$$R_{n_k+m} \mathfrak{h}(\underline{x}) - c(\underline{h}) = \int (R_{n_k} \mathfrak{h} - c(\underline{h}))\, \mathrm{d}\Theta^m_{n_k,\underline{x}}.$$

We choose $\underline{x}^{n_k+m}$ with $R_{n_k+m} \mathfrak{h}(\underline{x}^{n_k+m}) = c(\underline{h})$ and consider the measures

$$\Theta^m_{(n_k)} = \Theta^m_{n,\underline{x}^{n_k+m}}.$$

It follows that

$$0 = \int (R_{n_k} \mathfrak{h} - c(\underline{h}))\, \mathrm{d}\Theta^m_{(n_k)}$$

hence if Θ is any accumulation point of $\{\Theta^m_{(n_k)}\}$, then

$$\int (\mathfrak{g} - c(\underline{h}))\, \mathrm{d}\Theta = 0.$$

Observe that $\Theta(U) > 0$, and since $\mathfrak{g} - c(\underline{h}) \geq 0$, we deduce that there exists some $\underline{x}_U \in U$ such that $g(\underline{x}_U) = c(\underline{h})$. But U is arbitrary, hence $\mathfrak{g} \equiv c(\underline{h})$. We have shown:

Theorem 2.4. *For every $\mathfrak{h} \in \mathcal{C}(\Sigma_A)$ the family $\{R_n \mathfrak{h}\}_n$ converges uniformly to a constant $c(\underline{h})$.*

3 Equicontinuity of the conditional expectations

Recall that we are denoting by ξ the partition of $S(A, R)$ into local unstable manifolds,

$$\xi(\Psi([\underline{x}, t], \underline{y})) = \{\Psi([\underline{x}, t], \underline{y}')) : \underline{y}' \in \Sigma^+_A\}$$

and for $n \geq 0$ we denote by ξ^n the partition with atoms

$$\xi^n(\Psi([\underline{x}, t], \underline{y})) = \{\Psi([(x_k)_{k \leq n} a_{k+1} \cdots a_0, t], \underline{y}') : \underline{y}' \in \Sigma^+_A, A_{x_{-k} a_{k+1}} = 1, A_{a_j, a_{j+1}} = 1, k+1 \leq j < -1\}.$$

It holds

$$\xi^n(\Psi([\underline{x}, t], \underline{y})) = \bigsqcup_j \xi(\Psi([\underline{x}^j, t], \underline{y})) \tag{6}$$

for some points $\underline{x}^j \in \Sigma^-_A$.

We fix m a probability measure in $S(A,R)$ whose conditionals on the unstable sets are given by the family $\{\nu_p^u\}$. Given $\mathfrak{h}\in\mathcal{C}(\Sigma_A)$ the conditional expectation of $\mathfrak{h}$ with respect to ξ^n can be computed in the point $p=\Psi([\underline{x},t],\underline{y})$ as

$$E_n\mathfrak{h}(p)=\frac{1}{\nu_p^u(\xi^n(p))}\int_{\xi^n(p)}\mathfrak{h}\,\mathrm{d}\nu_p^u=\frac{1}{\nu_{\sigma^{-n}\underline{z}}^u(W^u_{\mathrm{loc}}(\sigma^{-n}\underline{z}))}\int_{W^u_{\mathrm{loc}}(\sigma^{-n}\underline{z})}\mathfrak{h}([\sigma^n\underline{w},t])\,\mathrm{d}\nu^u_{\sigma^{-n}\underline{z}}(\underline{w}).$$

where $\underline{z}=\underline{x}\cdot\underline{y}$

It now follows by Theorem 2.4 that $\{E_n\mathfrak{h}\}_n$ converges uniformly to some constant $\underline{c}(\mathfrak{h})$, and therefore

$$\int\mathfrak{h}\,\mathrm{dm}=\int E_n\mathfrak{h}\,\mathrm{dm}\xrightarrow[n\to\infty]{}\int c(\mathfrak{h})\,\mathrm{dm}=c(\mathfrak{h}),$$

which in turn implies $c(\mathfrak{h})=\int\mathfrak{h}\,\mathrm{dm}$. This shows that there is at most one measure having conditionals given by $\{\nu_x^u\}$, and since the equilibrium state for the potential ϕ satisfies this condition, it is this referred measure. This concludes the proof of the Main Theorem.

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