Franks' theorem, after A. Katok and F. Rodriguez-Hertz

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Abstract

These are notes on a lecture given by F. Rodriguez-Hertz in Trieste (2012), explaining an alternative proof of the well known Franks' result: any Anosov diffeomorphism in a surface is conjugate to a linear hyperbolic map acting in the torus.

In this notes we'll prove the following.

Theorem (Franks, 1971). *Let $f : M \to M$ be an Anosov diffeomorphism, where M is closed surface. Then M is the Torus $\mathbb{T}^2$ and f is conjugate to a hyperbolic linear map $A : \mathbb{T}^2 \to \mathbb{T}^2$.*

The proof that we'll present is due to A. Katok and F. Rodrigues-Hertz [2], where a more general version for hyperbolic actions is given. It is a variation of a proof given by Hiraide [1]. As usual, any mistake comes from my own lack of understanding.

Convention $f : M \to M$ denotes an Anosov diffeomorphism on a compact surface.

For the moment we won't rely on the fact that $M = \mathbb{T}^2$, which can be deduced from basic topology. To establish the Theorem we'll use the following properties of f.

M-1 There exist one dimensional f-invariant foliations $\mathcal{W}^s = \{W^s(x)\}_{x\in M}, \mathcal{W}^u = \{W^u(x)\}_{x\in M}$. Their leaves are homeomorphic to $\mathbb{R}$.

M-2 There exist families of σ-finite measures $\{\nu_x^s\}_{x\in M}, \{\nu_x^u\}_{x\in M}$ satisfying the following properties.

(a) ν_x^s, ν_x^u are non-atomic and locally finite . The measure ν_x^s is fully suppported on $W^s(x)$, while the measure ν_x^u is fully supported on $W^u(x)$.

(b) The maps $x \mapsto \nu_x^s, x \mapsto \nu_x^u$ are continuous (see below).

(c) $y \in W^s(x)$ then there exists $c^s(x,y) > 0$ such that $\nu_x^s = c^s(x,y)\nu_y^s$. Likewise if $y \in W^u(x)$ then there exists $c^u(x,y) > 0$ such that $\nu_x^u = c^u(x,y)\nu_y^u$.

(d) For every $x \in M$ there exists $\lambda^s(x) > 0$ such that $f_*\nu_x^s = \lambda^s(x)\nu_{fx}^s$.

M-3 For every $y \in W^u(x)$ there exists $\mathrm{hol}_{y,x}^u : W^s(x) \to W^s(y)$ the holonomy (transport) map such that

(a) $\mathrm{Dom}(\mathrm{hol}_{y,x}^u) = W^s(x)$.

(b) $(\mathrm{hol}_{y,x}^u)_*\nu_x^s = \nu_x^y$.

Given $x \in M$ we seek to define a "good" parametrization of $W^s(x)$, that is, a homeomorphism $H_x^s : \mathbb{R} \to W^s(x)$ that'll satisfy the following properties.

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H-1 $H^s_x(0) = x$.

H-2 $y \in W^s(x) \Rightarrow (H^s_y)^{-1} \circ H^s_x : \mathbb{R} \to \mathbb{R}$ is an affine map; there exists $\tilde{c}(y,x) \neq 0, \tilde{r}(y,x) \in \mathbb{R}$ such that

$$(H^s_y)^{-1} \circ H^s_x(t) = \tilde{c}(y,x) \cdot t + \tilde{r}(y,x).$$

H-3 $(H^s_{fx})^{-1} \circ f \circ H^s_x(t) = \tilde{\lambda}^s(x) \cdot t$, for some $\tilde{\lambda}^s(x) \neq 0$.

H-4 $y \in W^u(x)$ implies

$$(H^s_y)^{-1} \circ \mathrm{hol}^u_{y,x} \circ H^s_x(t) = \tilde{a}(y,x) \cdot t$$

for some $\tilde{a}(y,x) \neq 0$.

To this end, we pick a measurable orientation for $\mathcal{W}^s$ and for $x \in M, t \in \mathbb{R}$ we let $H^s_x(t)$ be the unique point in $W^s(x)$ such that

$$\nu_x([x, H^s_x(t)]) = |t|$$

That is, $(H^s_x)^{-1}(y) = \pm\nu_x([x,y])$ (depending on the orientation).

Exercise. Check using properties **M-1**, **M-2**, **M-3** that H^s_x is a well defined homeomorphism satisfying **H-1**, **H-2**, **H-3** and **H-4**.

Now we fix $x \in M$ and define $\hat{h}_x : W^s(x) \times W^u(x) \to M$ by

$$\widehat{h}_x(y,z) = \mathrm{hol}^s_{z,x}(y).$$

Observe that

- $\widehat{h}_x(x,x) = x$.
- $\widehat{h}_x(y,x) = y$
- $\widehat{h}_x(y,z) \subset W^u(y) \cap W^s(z)$.

Using $\widehat{h}_x$ we define $h_x : \mathbb{R} \times \mathbb{R} \to M$ as the composition

$$\begin{aligned}(t^s,t^u) \mapsto (H^s_x(t^s), H^u_x(t^u)) \mapsto \widehat{h}_x(H^s_x(t^s), H^s_x(t^u)) &= \mathrm{hol}^s_{H^u_x(t^u),x}(H^s_x(t^s)) \\ &= H^s_{H^u_x(t^u)}(\widetilde{a}(H^u_x(t^u))t^s, x)\end{aligned}$$

Let $\mathcal{A} := \{\text{affine maps of } \mathbb{R}^2, Lz = Az + b : A \text{ diagonal}\}$. It is direct to check (and well known) that $\mathcal{A}$ is a Lie group.

Lemma 0.1. *If $w^s, w^u \in \mathbb{R}$ are such that $h_x(w^s, w^u) = y$ then there exists $L \in \mathcal{A}$ satisfying*

1. $L(0,0) = (w^s, w^u)$.
2. $h_x \circ L = h_y$.

The proof is not hard. Now we consider $\Gamma \subset \mathcal{A}$ the subgroup given by

$$\Gamma := \{L \in \mathcal{A} : h_x \circ L = h_x\}$$

Lemma 0.2. *Γ acts discontinously on $\mathbb{R}^2$.*

Proof. Otherwise there would exist a sequence $\{L_n\} \subset \Gamma$ of distinct elements such that $L_n(0,0) \xrightarrow[n\mapsto\infty]{} (0,0)$. This is absurd by transversality of E^s_x, E^u_x. ■

We recall the following two facts of subgroups of the Möbius transformations group $\mathtt{PSl}_2(\mathbb{C})$:

- If $G < \mathtt{PSl}_2(\mathbb{C})$ acts discontinuously on a subset of $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then G is discrete.
- A subgroup of $\mathtt{PSl}_2(\mathbb{C})$ containing a map $az + b$, $a \neq 1$ cannot act discontinuously on $\mathbb{C}$.

Corollary 0.3. *Γ is a discrete group of translations.*

It follows by lemma 0.1 that h_x defines a bijective bi-measurable map

$$h : \mathbb{R}^2/\Gamma \to M$$

Let us assume for now that x is a fix point of f (in particular $\mathrm{Fix}(f) \neq \emptyset$) and write $h = h_x$. Consider A the map $A = h^{-1} \circ f \circ h$ and note that A is linear and sends Γ in Γ. We consider the manifold $X := \mathbb{R}^2/\Gamma$ and observe that we can induce μ a smooth measure (Haar in fact) on it. By construction, $h_*\mu$ is f-invariant and moreover it has conditionals equivalent to ν_y^s, ν_y^u almost everywhere. This implies that $h_*\mu$ is finite over compact sets, and thus finite. Hence $\Gamma \approx \mathbb{Z} \times \mathbb{Z}$, and therefore $X = \mathbb{T}^2$.

We have advanced as much as we could without assuming any structure on $\mathcal{W}^s, \mathcal{W}^u$. To finish, we need an extra argument: $\mathcal{W}^s, \mathcal{W}^u$ are orientable, hence h is in fact an homeomorphism. There are several ways to prove this, one can for example use that E^s, E^u are $\mathcal{C}^1$ (we are in dimension two) and invoke the following classical theorem.

Theorem 0.4. *Let E be a smooth direction field on a surface without periodic orbits. Then E is orientable.*

Remark 0.1. *It is easy to prove that f has a periodic point, therefore the previous argument shows that some power of it is conjugate to a linear hyperbolic map A. Since A is diagonalizable, it has roots of any order and it follows that f is conjugate to a linear hyperbolic map as well.*

References

[1] K. Hiraide. "A simple proof of the Franks–Newhouse theorem on codimension-one Anosov diffeomorphisms". In: *Ergodic Theory and Dynamical Systems* 21.03 (2001). DOI: 10.1017/s0143385701001390 (cited on page 1).

[2] A. Katok and F. Rodriguez Hertz. "Measure and cocycle rigidity for certain nonuniformly hyperbolic actions of higher-rank abelian groups". In: *Journal of Modern Dynamics* 4.3 (2010), pages 487–515. DOI: 10.3934/jmd.2010.4.487 (cited on page 1).