Hyperbolic Geometry

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CHAPTER 1

Hyperbolic Geometry

Recall that for $\alpha \in \mathbb{D}$ the map $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ defined by

$$\varphi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$$

is bi-holomorphic ($\varphi_\alpha^{-1} = \varphi_{-\alpha}$), sends α to 0 and 0 to $-\alpha$. Furthermore,

$$\varphi_\alpha'(z) = \frac{1-|z|^2}{(1-\bar{\alpha}z)^2}$$

and in particular

$$\varphi_\alpha'(0) = 1-|\alpha|^2$$
$$\varphi_\alpha'(\alpha) = \frac{1}{1-|\alpha|^2}.$$

Suppose that $f : \mathbb{D} \circlearrowleft$ is holomorphic and fix $\alpha \in \mathbb{D}$. The holomorphic function $h = \varphi_{f\alpha} \circ f \circ \varphi_{-\alpha} : \mathbb{D} \circlearrowleft$ fixes 0, therefore by Schwartz's lemma $|h'(0)| = |\varphi_{f\alpha}| \cdot |f'(\alpha)| \cdot |\varphi_{-\alpha}(0)| \leq 1$, that is

$$|f'(\alpha)| \leq \frac{1-|f(\alpha)|^2}{1-|\alpha|^2}.$$

There is equality if and only there exists $\lambda \in \mathbb{S}^1$ such that $f = \varphi_{-f\alpha} \circ m_\lambda \circ \varphi_\alpha$, where $m_\lambda(z) = \lambda \cdot z$; in this case $f \in \mathscr{Mob}$, therefore bijective.

In particular

$$|f'(0)| \leq 1-|f(0)|^2; \text{ if f is not bijective then } |f'(0)| < 1-|f(0)|^2.$$

Theorem 1.0.1 (Schwartz-Pick Lemma). *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $z \in \mathbb{D}$, then*

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}.$$

Equality implies that f is Möbius.

On $\mathbb{H}$ we consider the Riemannian metric

$$\mathrm{d}s^2_{\mathbb{H}} = \frac{|\,\mathrm{d}z|}{(\mathrm{Im}(z))^2};$$

that is, for $v, w \in \mathbb{R}^2$ and $z \in \mathbb{H}$ we compute the inner product of the vectors $v, w \in T_z\mathbb{H}$ as

$$\langle v, w\rangle = \frac{v \cdot w}{\mathrm{Im}\, z^2}.$$

where $v \cdot w$ is the Euclidean inner product between these two vectos. This metric is conformal, in particular

$$\boxed{\text{angles measured with } \mathrm{d}s_{\mathbb{H}} \text{ are the same as measured with the Euclidean metric.}}$$

Similarly, on $\mathbb{D}$ consider the Riemannian metric given by

$$\mathrm{d}s^2_{\mathbb{D}} = \frac{4|dz|^2}{(1-|z|^2)^2}$$

Definition 1.0.1. *The metric defined by $\mathrm{d}s_{\mathbb{H}}, \mathrm{d}s_{\mathbb{D}}$ is the Poincaré metric in the space $\mathbb{H}, \mathbb{D}$.*

The induced distance, called the Poincaré distance in $X = \mathbb{H}, \mathbb{D}$ is

$$\mathtt{d}(a, b) = \inf\{l(\gamma) : \gamma : [0, 1] \to X, \gamma(0) = a, \gamma(1) = b\}$$

where

$$l(\gamma) = \int_0^1 \mathrm{d}s_X(\gamma'(t))\,\mathrm{d}t.$$

exercise Show that $(\mathbb{H}, \mathrm{d}s_{\mathbb{H}}), (\mathbb{D}, \mathrm{d}s_{\mathbb{D}})$ are have constant sectional curvature $K_g = -1$.

Convention From now on, $X = \mathbb{D}, \mathbb{H}$ are always equipped with their Poincaré metric, unless explicitly stated, and every metric notion is referred to this metric. For example, if we say that $f : X \circlearrowleft$ is an isometry, we mean an isometry for the Poincaré metric, $f_*\,\mathrm{d}s_X = \mathrm{d}s_X$.

Recall:. *Let X be a Riemannian manifold (say, a surface equipped with a Riemannian metric). A curve $\gamma : I \to X$ is a geodesic if its derivative has constant norm c, and*

$$\forall t_0 \in I \exists \epsilon > 0 : t, s \in (t_0 - \epsilon, t_0 + \epsilon) \Rightarrow l(\gamma|[t, s]) = c|t - s|.$$

That is, if γ locally minimizes the distance among its points. If $f : X \circlearrowleft$ is an isometry and γ is a geodesic then clearly $f \circ \gamma$ is a geodesic. We denote

$$\mathrm{Isom}(X) = \{f : X \circlearrowleft : f\ \mathcal{C}^1 \textit{isometry}\}$$

and $\mathrm{Isom}^+(X) \subset \mathrm{Isom}(X)$ the subset of orientation preserving isometries.

Observe the following consequence of Schwartz-Pick.

Proposition 1.0.2. *$f : \mathbb{D} \circlearrowleft$ holomorphic, then*

$$\forall a, b \in \mathbb{D}, \mathtt{d}(fa, fb) \le \mathtt{d}(a, b).$$

There is $=$ for some pair $a \neq b$ if and only if $f \in \mathscr{M}ob$.

Proof. Considering the line element $\mathrm{d}s_{\mathbb{D}} = \frac{2|dz|}{1-|z|^2}$, we deduce directly by Schwartz-Pick

$$f^*\left(\frac{2|dz|}{1-|z|^2}\right) = \frac{2|f'(z)|\cdot|dz|}{1-|fz|^2} \leq \frac{2|dz|}{1-|z|^2}$$

which gives the inequality. Moreover, due to continuity we deduce that the the equality $\mathtt{d}(fa, fb) = \mathtt{d}(a, b)$ for some $a \neq b$ implies that at some z we have $2\frac{|f'(z)|}{1-|fz|^2} = \frac{2}{1-|z|^2}$ which in turn implies that f is in $\mathscr{Mob}$. The reciprocal follows from the discussion at the beginning of this part. ∎

Corollary 1.0.3. $\mathtt{Aut}(\mathbb{D}) = \mathrm{Isom}_P^+(\mathbb{D})$.

Proof. If $f \in \mathtt{Aut}(\mathbb{D})$, we've already seen that it is of the form $f(z) = e^{i\theta}\varphi_\alpha(z)$, for some $\theta \in \mathbb{R}, \alpha \in \mathbb{D}$; one checks directly that $z \mapsto e^{i\theta}$ and φ_ϕ preserve the Poincaré metric, therefore $\mathtt{Aut}(\mathbb{D}) \subset \mathrm{Isom}_P^+(\mathbb{D})$.

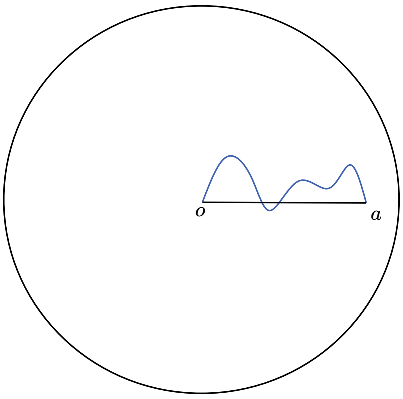
Conversely, if $f : \mathbb{D} \circlearrowleft$ is an orientation preserving isometry, then f is conformal (since $\mathrm{d}s_P$ is conformal), $\mathcal{C}^1$ and preserves orientation. Therefore is holomorphic, and by the previous proposition it is a Möbius transformation. ∎

Since $z \mapsto \bar{z}$ is an orientation reverse involution, we get:

Corollary 1.0.4. $\mathrm{Isom}_P(\mathbb{D}) = \mathtt{span}\{\mathrm{SU}(1,1), z \mapsto \bar{z}\}$.

Here is an important application.

Geodesics in $\mathbb{D}$ Let $a \in \mathbb{D} \cap \mathbb{R}_{>0}, a = x + i0$ and consider the curve $\gamma(t) = tx$ and any other curve $\tilde{\gamma}$ in $\mathbb{D}$ satisfying $\tilde{\gamma}(0) = 0, \tilde{\gamma}(1) = a$.



Write $\tilde{\gamma}(t) = u(t) + iv(t)$ and compute its length

$$l(\tilde{\gamma}) = \int_0^1 \frac{2\sqrt{\dot{u}^2+\dot{v}^2}}{1-(u^2+v^2)}\,\mathrm{d}t \geq \int_0^1 \frac{2|\dot{u}|}{1-u^2}\,\mathrm{d}t = 2\int_0^x \frac{du}{1-u^2} = \log(\frac{1+x}{1-x}) = l(\gamma).$$

We deduce that γ minimizes the distance between 0 and a, and by the same computation, given two points in $\mathbb{D} \cap \mathbb{R}_{>0}$ the horizontal segment minimizes the distance between them. We deduce that γ is a geodesic and $\mathtt{d}(0, a) = \log(\frac{1+x}{1-x})$.

Now if $a, b \in \mathbb{D}$ are any pair of points we can compute

$$\begin{aligned}\mathtt{d}(a,b) &= \mathtt{d}(\varphi_a(a), \varphi_a(b)) = \mathtt{d}(0, \varphi_a(b)) = \mathtt{d}(0, |\varphi_a(b)|)\\ &= \log \frac{1 + \left|\frac{b-a}{1-\bar{a}b}\right|}{1 - \left|\frac{b-a}{1-\bar{a}b}\right|} = \log\left(\frac{|1-\bar{a}b| + |b-a|}{|1-\bar{a}b| - |b-a|}\right).\end{aligned}$$

1.1 Geodesics in $\mathbb{H}$

Consider the Cayley transform $T : \mathbb{H} \to \mathbb{D}$,

$$T(z) = \frac{z-i}{z+i}.$$

As $T'(z) = \frac{2i}{(z+i)^2}$, the get

$$T^* \, \mathrm{d}s_{\mathbb{D}}|_{Tz} = \frac{2}{|z+i|^2} \frac{2|dz|}{1 - \left|\frac{z-i}{z+i}\right|^2} = \frac{4|dz|}{|z+i|^2 - |z-i|^2} = \frac{|dz|}{\mathrm{Re}(-iz)} = \frac{|dz|}{\mathrm{Im}(z)} = \mathrm{d}s_{\mathbb{H}}.$$

That is, $T : \mathbb{D} \to \mathbb{H}$ is an isometry.

Let us now compute the hyperbolic distance in $\mathbb{H}$. We start considering the particular case $a = iy, b = i$ and observe that

$$\mathtt{d}_{\mathbb{H}}(a, i) = \mathtt{d}_{\mathbb{D}}(T(a), 0) = \log \frac{1+r}{1-r}, \quad r = |T(a)|.$$

Since $r = \frac{|x-1|}{x+1}$, we get

$$\frac{1+r}{1-r} = \frac{x+1+|x-1|}{x+1-|x-1|} = \begin{cases} x & x \geq 1 \\ \dfrac{1}{x} & x < 1 \end{cases}$$

hence

$$\mathtt{d}_{\mathbb{H}}(a, i) = |\log x|.$$

Next suppose that $b = iy'$ and consider the isometry $f(z) = \frac{z}{y'}$; we get

$$\mathtt{d}_{\mathbb{H}}(a, b) = \mathtt{d}_{\mathbb{H}}(f(a), i) = |\log \frac{y}{y'}|.$$

We also note that since $\mathtt{d}_{\mathbb{H}}(\cdot, \cdot)$ is invariant under translations,

$$a = x + iy, b = x + iy' \Rightarrow \mathtt{d}_{\mathbb{H}}(a, b) = |\log \frac{y}{y'}|$$

The general case can be treated similarly.

We now use the (transitive) action $\mathtt{PSl}_2(\mathbb{R}) \curvearrowright \mathcal{N}_{hyp}$ and conclude that

$$\mathcal{N}_{hyp} \subset \{\text{traces of geodesics of } \mathbb{H}\}$$

In fact, those sets are equal.

Theorem 1.1.1. $\mathcal{N}_{hyp} = \{$*traces of geodesics of* $\mathbb{H}\}$.

Proof. Denote by $\gamma_{p,v}$ the geodesic determined by $(p,v) \in T\mathbb{H}$; it is no loss of generality to restrict ourselves to the case $|v| = 1$. Take one of such geodesics and consider the non-euclidean line L passing through p and tangent to v. Observe that L is well defined: if v is vertical this is obvious, otherwise consider the straight line which passes through p and is perpendicular to v, and let O be the point of intersection of this line with the x-axis. The semicircle centered at O with radius $|O - p|$ is the aforementioned L.

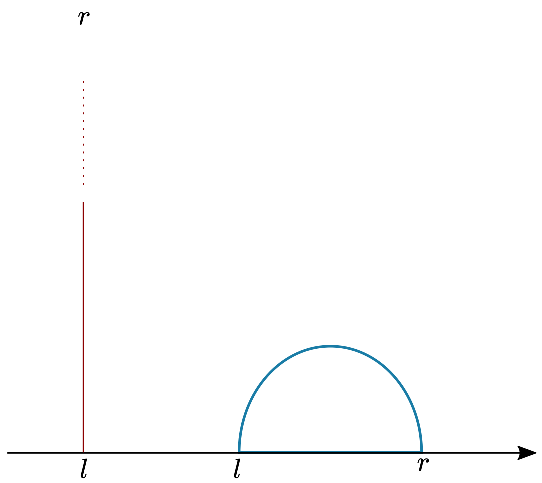


Figure 1.1: Possible non-euclidean lines.

Consider the Möbius transformation M sending $l(L) \mapsto 0, p \mapsto i, r(l) \to \infty$; necessarily M sends L to the vertical axis, whereas $M(\mathbb{R})$ is a line passing trough 0 that is perpendicular to $\vec{oy}$. It follows that $M(\mathbb{R}) = \mathbb{R}$ and $M = M_A$ for some $A \in \mathtt{PSl}_2(\mathbb{R})$.

We know that M is an isometry, and in particular $M(\gamma_{i,i})$ is the geodesic passing through p with tangent vector $M'(p)$. But note that $M(\gamma_{i,i})$ is a parametrization of L (with unit speed), hence $M'(p)$ is the tangent to L at z, i.e. $M'(p) = v$. This shows that $M(\gamma_{i,i}) = \gamma_{z,p}$, and in particular $\gamma_{z,p}$ is a parametrization of L. ∎

Remark 1.1.1. *The following picture contains an important historical fact.*

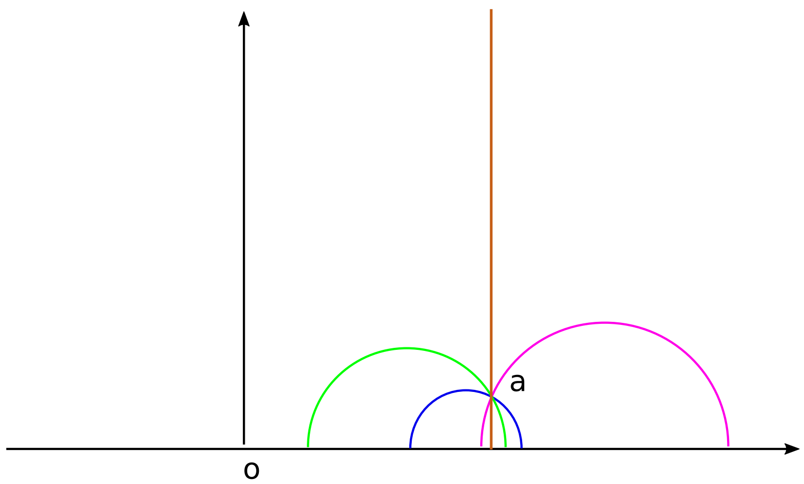


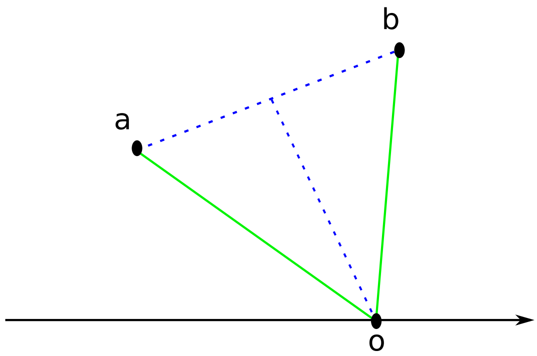
Figure 1.2: Infinitely many parallel "lines" to $\vec{oy}$ through the point a.

The 5th postulate of Euclides does not hold in the geometry model $(\mathbb{H}, \mathrm{d}s_P)$.

A similar argument shows the following.

Proposition 1.1.2. *Given* $a \neq b \in \mathbb{H}$ *there exists a unique geodesic*[1] $\gamma : \mathbb{R} \to \mathbb{H}$ *such that* $\gamma(0) = a, \gamma(1) = b$.

Proof. Without loss of generality a, b are not on the same vertical line (otherwise the result is direct). Let C be the semi-circle containing a, b and denote by $l < r$ its intersection with $\mathbb{R}$.



Define $f(z) = \frac{z-r}{z-l}$, and observe that $f \in \mathtt{Aut}(\mathbb{H})$ satisfies $f(r) = 0, f(l) = \infty$, therefore $f(C) = \vec{oy}$. Since $\vec{oy}$ is the unique geodesic between $f(a), f(b)$, the result follows. ■

During the proof of the previous theorem we have also shown that the action $\mathtt{PSl}_2(\mathbb{R}) \curvearrowright T_1\mathbb{H} = \mathbb{H} \times \mathbb{S}^1$ given by

$$A \cdot (z, v) = (M_A(z), M_A'(z)v)$$

is transitive. We readily compute the stabilizer of (i, i):

1. $\frac{ai+b}{ci+d} = i \Rightarrow a = d, b = -c$.
2. $\frac{1}{(ci+d)^2} i = i \Rightarrow -c^2 + d^2 + 2cdi = 1 \Rightarrow a^2 - b^2 = 1, ab = 0$.

Thus $b = c = 0, a = d = 1$, and the stabilizer is just the identity. By the orbit-stabilizer theorem we conclude.

Proposition 1.1.3. *There exists a smooth* $\mathtt{PSl}_2(\mathbb{R})$*-equivariant*[2] *identification* $T_1\mathbb{H} \approx \mathtt{PSl}_2(\mathbb{R})$. *A point* $(z, v) \in T_1\mathbb{H}$ *is identified with the matrix* A *such that* $M_A(i) = z, M_A'(i) = v$.

We have remarked that T sends $\partial\mathbb{H} = \mathbb{R}$ to $\partial\mathbb{D} = \mathbb{S}^1$; these are called the *boundaries at* ∞. Let $a = x + iy, b = x + i\epsilon$; by direct computation we get

$$\lim_{\epsilon \to \infty} \mathtt{d}_{\mathbb{H}}(a, b_\epsilon) = \lim_{\epsilon \to 0} |\log \frac{\epsilon}{y}| = \infty.$$

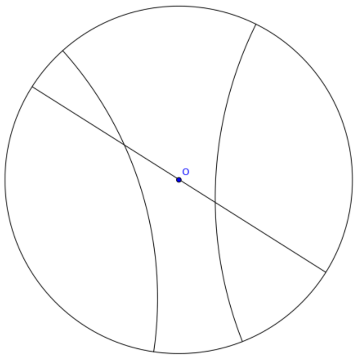
That is, $\mathtt{d}_{\mathbb{H}}(a, \partial\mathbb{H}) = \infty$, and likewise $\mathtt{d}_{\mathbb{H}}(a, \partial\mathbb{D}) = \infty$.

[1] Here we don't insists on γ' having unit norm.
[2] Thus, the action $\mathtt{PSl}_2(\mathbb{R}) \curvearrowright T_1\mathbb{H}$ corresponds to left matrix multiplication.

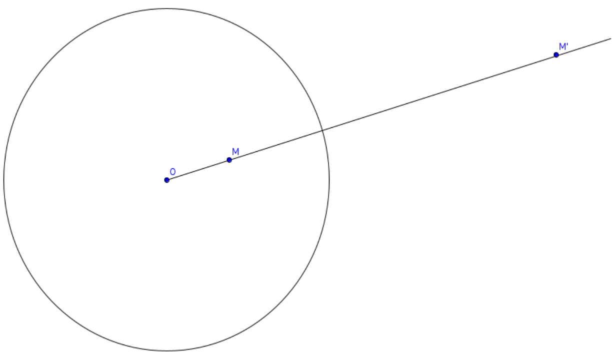
1.2 Geodesics in the disc model

Since the Cayley transform is an isometry between $\mathbb{H}$ and $\mathbb{D}$ sending $\mathbb{R}$ to $\mathbb{S}^1$, and since Möbius transformations sends lines/circles into lines/circles while preserving angles, we imediately deduce the following.

Corollary 1.2.1. *The traces of geodesics in $\mathbb{D}$ are the curves $\mathbb{D} \cap T$ where T is a circle $\perp \partial\mathbb{D}$.*



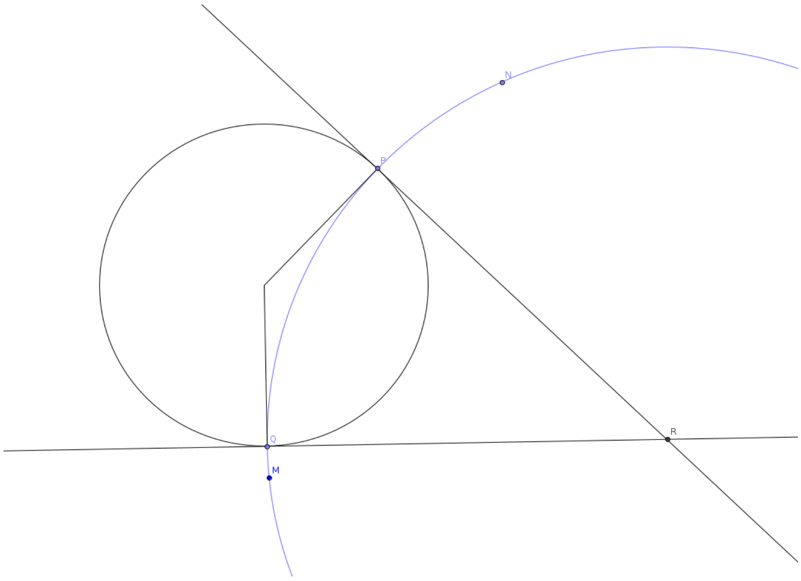
It is instructive however to make a direct approach. Define $I : \mathbb{D} \circlearrowleft$ the geometrical inversion, $I(z) = \frac{1}{\bar{z}}$.



Clearly I preserves angles, and if $C \subset \hat{\mathbb{C}}$ is a circle/line then $I(C)$ is also a circle/line.

Claim. *Given $P, Q \in \mathbb{S}^1$ there exists a unique circle C passing through P, Q that is orthogonal to $\mathbb{S}^1$.*

Consider the picture below.



If $\{P, Q\} = C \cap \mathbb{S}^1$ and $C \perp \mathbb{S}^1$ then necessarily the center (R) of C is in the intersection of the tangents to $\mathbb{S}^1$ through P, Q, and the radius of C is $|P - R|$. This shows uniqueness, also gives a recipe to construct C.

Proposition 1.2.2. *Let $C \subset \mathbb{C}$ be a circle. Then C is orthogonal to $\mathbb{S}^1 \Leftrightarrow I(C) = C$.*

Proof. Let $\{P, Q\} = \mathbb{S}^1 \cap C$.

$\Rightarrow$ Then $I(C)$ is a circle orthogonal to C through P, Q thus by uniqueness of such circle, $I(C) = C$.

$\Leftarrow$ Assume that $\angle_P(C, \mathbb{S}^1) < \frac{\pi}{2}$. Since I inverts the sense of the angles, we see that C cannot be fixed, as $I(P) = P$.

■

Now take $M, N \in \mathbb{D}$ and denote $M' = I(M), N' = I(N)$. Consider C the (unique) through M, N, M' (or equivalently, through M, N, N').

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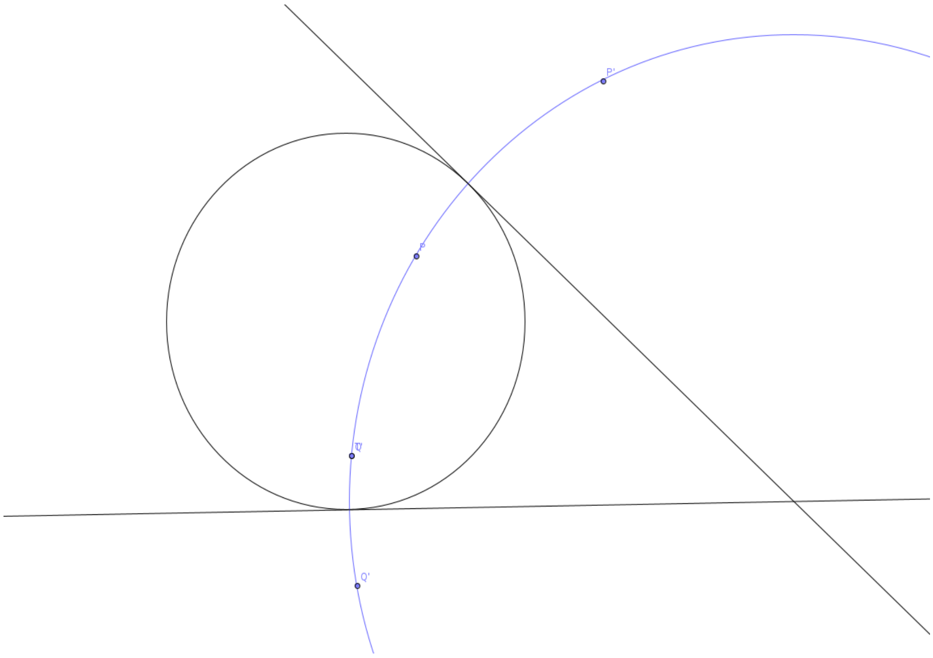

Then C is clearly fixed under I, therefore $C \perp \mathbb{S}^1$. Denote $\tilde{C} = C \perp \mathbb{S}^1$ and observe that $\varphi_P(\tilde{C})$ is a circle/line perpendicular to $\mathbb{S}^1$ joining $\varphi_P(P) = 0$ with $\varphi_P Q$; therefore $\varphi_P(\tilde{C})$ is a diameter in $\mathbb{D}$, and in particular is the unique (traze of) geodesic these two points. We conclude that $\tilde{C}$ is the traze of the unique geodesic joining P with Q.

1.3 Hyperbolic circles

Consider the hyperbolic circle in $\mathbb{D}$ of center 0 and radius $r > 0$,

$$C = \{z \in \mathbb{D} : \mathtt{d}(0, z) = r\} = \{z \in \mathbb{D} : \log \frac{1 + |z|}{1 - |z|} = r\}.$$

Note that

$$\log \frac{1 + |z|}{1 - |z|} = r \Leftrightarrow |z| = \frac{e^r - 1}{e^r + 1} = \tanh(\frac{r}{2}).$$

We deduce that C coincides with the euclidean circle of center 0 and radius $\tanh(\frac{r}{2})$.

Likewise

$$D_P(0, r) = D_{\text{Euc}}(0, \tanh(\frac{r}{2})).$$

Now consider an arbitrary hyperbolic circle $C = C_P(z_0, r)$, and apply the map φ_{z_0}; we get

$$\varphi_{z_0}(C) = C_P(0, r) = C_{\text{Euc}}(0, \tanh(\frac{r}{2}))$$

We deduce that C is an Euclidean circle of different radius and typically different center (if $z_0 \neq 0$).

Corollary 1.3.1. $(\mathbb{D}, \mathtt{d}_P)$ *is complete.*

As consequence of the Hopf-Rinow theorem, $K \subset \mathbb{D}$ is compact if and only if is closed and bounded. Similarly for $\mathbb{H}$.

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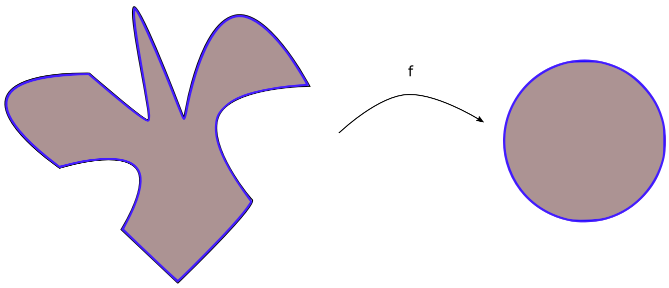
CHAPTER 2

The uniformization theorem

Here we'll prove the followin important theorem.

Theorem 2.0.1. *Let $U \subset \mathbb{C}$ be simply connected domain, $U \neq \mathbb{C}$. Then there exists $F : U \to \mathbb{D}$ bi-holomorphic.*

In other words any simply connected domain in $\mathbb{C}$ which is not the whole plane is conformally equivalent to the disc. Note that $\mathbb{C}$ is not holomorphically equivalent to $\mathbb{D}$ (by Liouville's theorem).



The proof is "surprinsingly simple[1]".

Lemma 2.0.2. *If $U \subsetneq \mathbb{D}$ is a simply connected domain then there exists an holomorphic embedding $F : U \to \mathbb{D}$ such that*

$$a, b \in U, a \neq b \Rightarrow \mathtt{d}_{\mathbb{D}}(fa, fb) > \mathtt{d}_{\mathbb{D}}(a, b).$$

Proof. Take $\alpha \in \mathbb{D} \setminus U$ and note that since $\varphi_\alpha : \mathbb{D} \circlearrowleft$ does not have any zeros in the simply connected domain U, there exists $g \in H(U)$ such that $g^2 = \varphi_\alpha$. We claim that g is injective: indeed, if $g(a) = g(b)$ then $\varphi_\alpha(a) = \varphi_\alpha(b)$ and $a = b$.

Let $h(z) = z^2, h : \mathbb{D} \circlearrowleft$ contracts the Poincaré metric, and since $h \circ g = \varphi_\alpha$ is an isometry, necessarily g expands the Poincaré metric. ■

Proof of the Uniformization Theorem. Consider the family

$$\mathcal{F} = \{f : U \to \mathbb{D} : f \in H(U), f \text{ 1-1 (embedding)}\}.$$

[1]According to Thurston...

Note that for every $f \in \mathcal{F}$, the set $f(U) \subset \mathbb{D}$ is a simply connected domain.

Claim: $\mathcal{F} \neq \emptyset$.

Choose $p \notin U$ and define $h : U \to \mathbb{C}, h(z) = z - p$; h is holomorphic in U and without zeros, therefore there exists $g \in H(U)$ with $g^2 = h$. Arguing as in the lemma we see that

$$a \neq b \Rightarrow g(a) \neq \pm g(b).$$

The set $V = g(U)$ is a domain, and $w \in V$ implies that $-w \notin V$. Take $q \in V, q \neq 0$ and choose $0 < r < |q|$ such that $\mathbb{D}(q, r) \subset V$.

Claim: $\mathbb{D}(-q, r) \cap V = \emptyset$.

Otherwise if $w \in \mathbb{D}(-q, r) \cap V$, then $|w + q| < r \Rightarrow |q - (-w)| < r \Rightarrow -w \in V$, which is not true.

We can then define $f : U \to \mathbb{C}$ by

$$f(z) = \frac{r}{g(z) + q}$$

and note that f is holomorphic, injective and $|f| < 1$, hence $f \in \mathcal{F}$.

Next we fix $q \in U$ and let $M = \sup\{|f'(q)| : f \in F\}$; since $\mathcal{F} \neq \emptyset$ we have $M > 0$, and by Schwartz pick necessarily $M < \infty$. Note also that $\mathcal{F}$ is normal, because it is a uniformly bounded family of holomorphic functions (Montel's theorem).

Choose $(f_n)_n \subset \mathcal{F}$ with $\lim_n |f_n'(q)| = M$, and without loss of generality we can assume that $\|f_n - f\|_{C^0} \xrightarrow[n\to\infty]{} 0$, for some $f \in H(U)$. By Hurewicz' theorem we know that f is either injective or constant, but since $|f'(q)| = M > 0$, necessarily f is injective.

As $f_n(U) \subset \mathbb{D} \forall n \Rightarrow f(U) \subset \mathtt{cl}(\mathbb{D})$; on the other hand f is open, and therefore $f(U) \subset \mathbb{D}$. We have shown that $f \in \mathcal{F}$.

Claim: $f(U) = \mathbb{D}$.

Otherwise, using the lemma we could find $g : f(U) \to \mathbb{D}$ injective that expands the Poincaré metric. But then

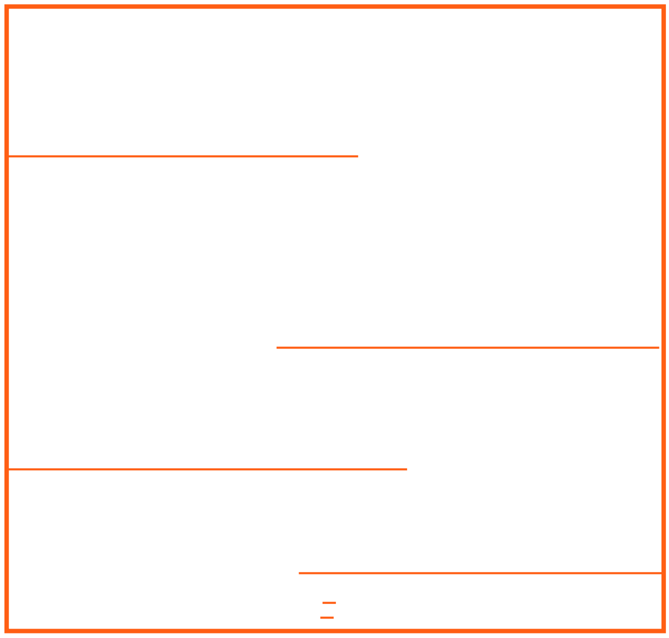
$$\|g'(fq) \cdot f'(q)\|_P = M \cdot \|g'(fq)\|_P > M$$

which is absurd since $g \circ f \in \mathcal{F}$.

We have thus found $f : U \to \mathbb{D}$ bi-holomorphic. ■

Question. *Consider $U \neq \mathbb{C}$ simply connected domain, and $f : U \to \mathbb{D}$ bi-holomorphic. Can we extend f to an homeomorphism $F : \mathtt{cl}(U) \to \mathtt{cl}(\mathbb{D})$?*

Not in general; the boundary ∂U could be very complicated and not a circle.



However the following is known.

Theorem 2.0.3 (Carathéodory)**.** *Let $f : U \to \mathbb{D}$ be a conformal map, $U \subset \mathbb{C}$ simply connected. Then f extends to an homeomorphism $F : \mathtt{cl}(U) \to \mathtt{cl}(\mathbb{D})$ if and only if ∂U is a Jordan curve.*