# Shadowing of pseudo-orbits

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March 15, 2023

**Abstract**

In this note we give an effective version of the Hirsch-Pugh-Shub shadowing theorem.

## 1 Partially hyperbolic systems

We will be concerned with a generalization of the classical shadowing property for hyperbolic systems. The setting for this is in the context of partially hyperbolic diffeomorphisms, which we now define.

**Definition 1.1.** *Let $M$ be a closed Riemannian manifold. We say that a $\mathcal{C}^r$ diffeomorphism $f : M \to M$ is partially hyperbolic if there exists a continuous splitting of the tangent bundle into a Whitney sum of the form*

$$TM = E^u \oplus E^c \oplus E^s$$

*where neither of the bundles $E^s$ nor $E^u$ are trivial, and such that*

1. *all bundles $E^u, E^s, E^c$ are $Df$-invariant;*
2. *for all $x \in M$ and for all unit vectors $v^\sigma \in E^\sigma_x$ $(\sigma = s, u, c)$,*
   (a) $\|D_x f(v^s)\| < \|D_x f(v^c)\| < \|D_x f(v^u)\|$;
   (b) $\|D_x f(v^s)\| < 1 < \|D_x f(v^u)\|$.

**Remark 1.1.** *It is no loss of generality to assume that the Riemannian metric is such that the bundles $E^s, E^c, E^u$ are orthogonal, and we will assume that from now on.*

The bundles $E^s, E^u, E^c$ are the *stable, unstable* and *center* bundle respectively. We also define the bundles $E^{cu} = E^c \oplus E^u$ and $E^{cs} = E^s \oplus E^c$, the *center stable* and *center unstable* bundles. Some important examples of partially hyperbolic diffeomorphisms are ergodic automorphisms of the torus, time-$1$ maps of hyperbolic flows, circle extensions over Anosov diffeomorphisms, and perturbations of those. For a discussion of these and other examples we refer to [5].

The stable manifold theorem (see [8]) implies that the bundles $E^u$ and $E^s$ are integrable to continuous foliations $\mathcal{W}^u, \mathcal{W}^s$ whose leaves are of the same degree of differentiability than $f$. These leaves are homeomorphic to Euclidean spaces of the corresponding dimension. Nonetheless, the transversal regularity of those foliations is only Hölder in general [4]. The integrability of the center bundle $E^c$, on the other hand, cannot be asserted in general as the example in [7] shows (see also [3]). Establishing necessary and sufficient conditions that guarantee this property remains one of most important problems in the area.

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Some partial results in this matter can be found in [6],[2] and [1]. Let us point out also that for most known examples the center bundle is integrable.

Here however, we are interested in the properties of the foliations that integrate the bundles $E^c, E^{cs}, E^{cu}$. We will work then with systems satisfying the following.

**Definition 1.2.** *A partially hyperbolic diffeomorphism is* dynamically coherent *if the bundles $E^c, E^{cu}$ and $E^{cs}$ are integrable to $\mathcal{C}^{1,0}$ invariant foliations*[1] *$\mathcal{W}^c, \mathcal{W}^{cu}, \mathcal{W}^{cs}$ and such that*

$$\mathcal{W}^c = \{W^{cu} \cap W^{cs} : W^{cu} \in \mathcal{W}^{cu}, W^{cs} \in \mathcal{W}^{cs}\}.$$

As explained in [2], it follows that

1. $\mathcal{W}^s$ sub-foliates $\mathcal{W}^{cs}$
2. $\mathcal{W}^u$ sub-foliates $\mathcal{W}^{cu}$

**Notation:** for $r > 0, x \in M, \sigma \in \{s, c, u, cs, cu\}$ denote $W^\sigma(x, r)$ the plaque centered at$x$ of radius $r$ of the foliation $W^\sigma$.

**Local Product Structure.** Dynamical coherence implies that the system has *local product structure*, namely there exists some $c_{lps} > 0$ such that if $d(x, y) < c_{lps}$ and $P_x, P_y$ denote plaques of $\mathcal{W}^c$ centered at $x, y$ of radius $c_{lps}$ then $W^s(P_x, c_{lps})$ meets $W^u(P_y, c_{lps})$ along a plaque of $\mathcal{W}^c$ of radius at least $c_{lps}/2$. In fact, local product structure is equivalent to dynamical coherence; the proof is not hard.

If the system has local product structure one can specify locally each plaque as follows. For a point $x \in M$ we define the sets $H_x = W^u(P_x, 2c_{lps}), V_x = W^s(P_x, 2c_{lps})$. Given any two points $x, y \in M$ such that $d(x, y) < c_{lps}$ both intersections $H_x \cap V_y$ and $V_x \cap H_y$ consist of center plaques, and by reducing $c_{lps}$ if necessary, there will be only one center plaque in the intersection $H_x \cap V_y$ meeting $W^u(x, 2c_{lps})$, and likewise there exists only one center plaque of $V_x \cap H_y$ intersecting $W^s(x, 2c_{lps})$. One sees then that the plaque through $y$ is specified by these two points in $W^u(x, 2\eta)$ and $W^s(x, 2c_{lps})$.

It is no loss of generality to assume that $c_{lps}$ is less than the injectivity radius of $\exp : TM \to M$, and by reducing this constant even more (using that $D_0 \exp = \mathrm{Id}$), that for $x, y \in M, d(x, y) < c_{lps}$ implies

$$d(x, y) = \inf\{\text{length}(\gamma) : \gamma : [0, 1] \to M \text{ piecewise differentiable}, \dot{\gamma} \in E^s \cup E^c \cup E^u\}.$$

Given $x, y \in M$ such that $d(x, y) < \delta$, where $0 \leq \delta \leq c_{lps}$ we define the *bracket between $x$ and $y$* as

$$[x, y] = W^s(x, 2c_{lps}) \cap W^{cu}(y, 2c_{lps})$$

Since $\exp| : \{(x, v) \in TM : \|v\| \leq c_{lps}\} \to M$ is nearly an isometry and since the angle between $E^c, E^u$ and $E^c, E^s$ is bounded away from zero, one sees that for some constant $A > 0$ depending only on the angles we have

$$d(x, [x, y]) < A\delta \tag{1}$$
$$d(y, [x, y]) < A\delta \tag{2}$$

**Holonomy.** Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism and fix once and for all a differentiable subbundle $N \subset TM$ transverse to $E^c$, with small angle with $E^s \oplus E^u$. For $0 < r \leq c_{lps}, x \in M$ denote $D^{su}(x, r) = \exp(\{(x, v) : \|v\| < r, v \in N^{su}\})$. Then for $x, y$ such that $y \in W^c(y, \frac{c_{lps}}{2})$, the center holonomy $H^c_{y,x} : D^{su}(x, \frac{c_{lps}}{2}) \to D^{su}(y, c_{lps})$ can be defined: for $z \in D^{su}(x, r)$

$$H^c_{y,x}(z) = W^c(z, c_{lps}) \cap D^{su}(y, c_{lps}).$$

We will also consider $H^c_{y,x}|W^s(x, \frac{c_{lps}}{2}) \to W^s(y, c_{lps})$ (if $y \in W^{cs}(x, \frac{c_{lps}}{2})$) and $H^c_{y,x}|W^u(x, \frac{c_{lps}}{2}) \to W^u(y, c_{lps})$ (if $y \in W^{cu}(x, \frac{c_{lps}}{2})$).

The foliations $\mathcal{W}^s, \mathcal{W}^u$ also (of course) induce related holonomy maps $H^s, H^u$. For the domain of $H^s$ we use discs inside center-unstable leaves, while for $H^u$ we use center-stable discs. We omit the discussion, which is analogous to what we wrote above.

[1] This means that the regularity of the leaves is $\mathcal{C}^1$ while the transversal regularity is continuous.

## 2 Pseudo-orbits

We recall some definitions.

**Definition 2.1.** *Let $f : M \to M$ be a homeomorphism preserving a foliation $\mathcal{F}$.*

1. *A sequence $\underline{x} = \{x_n : N_1 \leq N \leq N_2\}$ where $N_1 \in \mathbb{Z} \cup \{-\infty\}, N_2 \in \mathbb{Z} \cup \{\infty\}$ is called a $\delta$-pseudo-orbit for $f$ if $d(fx_n, x_{n+1}) \leq \delta$ for every $n = N_1, \ldots, N_2 - 1$.*
2. *The pseudo-orbit$\underline{y} = \{y_n : N_1 \leq N \leq N_2\}$ $\epsilon$-shadows the pseudo-orbit $\underline{x} = \{x_n : N_1 \leq N \leq N_2\}$ if $d(x_n, y_n) < \epsilon$ for every $n = N_1, \ldots, N_2$.*
3. *We say that the $\delta$-pseudo-orbit $\underline{x}$ is* subordinated to the foliation *$\mathcal{F}$ if for every $n \in \{N_1, \ldots, N_2 - 1\}$, $f(x_n)$ belongs the the plaque of $\mathcal{F}$ tha is centered at $x_{n+1}$ and has radius $\delta$.*

Now we fix a $f : M \to M$ a dynamically coherent partially hyperbolic diffeomorphism.

**Definition 2.2.** *Let $R > 0$ and let $\underline{x} = (x_n : N_1 \leq n \leq N_2)$ be a pseudo-orbit. We say that $\underline{x}$ is a $R$-center pseudo-orbit if it is a $\delta$-pseudo-orbit subordinated to $\mathcal{W}^c$.*

The first relevant theorem involving pseudo-orbitsof these notes is the following.

**Theorem A** (Shadowing I)**.** *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism. Then there exists a constant $C(f) > 0$ (only depending of $f$) such that for $R$ sufficiently small, any $R$-pseudo-orbit can be $C(f)R$-shadowed by a $C(f)R$-center pseudo-orbit.*

This result is due to Hirsch, Pugh and Shub (see theorem 7A-2 in [8]), where it is proved that given $\epsilon > 0$ there exists $R > 0$ such that any $R$-pseudo-orbit can be $\epsilon$-shadowed by aa $\epsilon$-pseudo-orbit subordinate to the foliation $\mathcal{W}^c$. Here we make explicit the relation between $R$ and $\epsilon$.

Observe that in the theorem above, if we assume that $\underline{x}$ is close to be a center pseudo-orbit, we expect to be able to shadow it with a center pseudo-orbit $\underline{y}$ with less error in the transverse direction. Let us formalize this notion.

**Definition 2.3.** *Let $0 < \delta \leq R$ and let $\underline{x} = (x_n : N_1 \leq n \leq N_2)$ be a pseudo-orbit. We say that $\underline{x}$ is a $(R, \delta)$-quasecenter pseudo-orbit if it is a $R$-pseudo-orbit and*

$$d\left(x_{n+1}, W^s(x_{n+1}, c_{lps}) \cap W^{cu}(fx_n, R)\right), d\left(x_{n+1}, W^u(x_{n+1}, c_{lps}) \cap W^{cs}(fx_n, R)\right) < \delta,$$

*for all $N_1 \leq n < N_2$.*

**Remark 2.1.** *The definition of quasecenter pseudo-orbit was suggested by J. Correa.*

We can prove the following.

**Theorem B** (Shadowing II)**.** *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism, and assume that its center bundle is $C^1$. Then there exist constants $R_0, C(f) > 0$ so that for each $0 < R \leq R_0$ we can find $0 < \delta_R < R$ and a continuous function $D_R : (0, \delta_R] \to [1, oo]$ verifying*

1. *$D_R(\delta) \xrightarrow[\delta \to 0]{} 1$.*
2. *Any $(R, \delta)$-quasecenter pseudo-orbit can be $C(f)\delta$-shadowed by a $D_R(\delta)R$-center pseudo-orbit.*

One should be able to prove a more general version of the above without assuming differentiability of the center bundle, replacing this condition by Hölder continuity of thge the center holonomies; possibly, it'll be necessary to relinquish the precise control given above. As these are expository notes, and since several interesting systems satisfy the cited conditions (for example, regular elements in Anosov actions), I won't worry about the general case.

# 3 Proof of the Theorem A

Denote

$$\lambda^s := \max\{\|D_x f|E^s\| : x \in M\}$$

and recall the definition of $A$ given in (1). Due the orthogonality of the bundles it follows that $A \xrightarrow[c_{lps}\to 0]{} 1$, thus by reducing $c_{lps}$ it is no loss of generality to assume that $r = A\lambda^s < 1$. Let $L = \max\{\mathrm{Lip}(f), \mathrm{Lip}(f^{-1})\}$ and $0 < \eta < c_{lps}$ so that $2\eta\left(\frac{AL}{1-r}\right)^2 < c_{lps}$. We consider only $0 < R < \eta$.

First assume that we have a finite $R$-center pseudo-orbit $\underline{x} = \{x_0, \dots, x_N\}$, and define the points $y_0, \dots, y_N$ by

- $y_0 = x_0$
- $y_n = [x_n, f(y_{n-1})]$ for $n = 1, \dots, N$.

Observe that $d(x_1, y_1), d(f(y_0, y_1) < AR$. We want to estimate $d(x_n, y_n)$; suppose then that we have proved that $d(x_{n-1}, y_{n-1}) < A\delta(1 + r + \cdots r^{n-1})$, with $y_{n-1} \in W^s(x_{n-1})$. Then we get

$$\begin{aligned} d(f(y_{n-1}), x_n) &\leq d(f(y_{n-1}), f(x_{n-1})) + d(f(x_{n-1}), x_n) \\ &\leq \lambda^s d(y_{n-1}, x_{n-1}) + R \leq A\lambda^s R(1 + r + \cdots r^{n-1}) + R < R(1 + r + \cdots r^n), \end{aligned}$$

and hence by (1)

$$d(x_n, y_n) = d(x_n, [x_n, f(y_{n-1})]) \leq Ad(x_n, f(y_{n-1})) < AR(1 + r + \cdots r^n) < \frac{A}{1-r}R$$
$$d(f(y_{n-1}), y_n) < \frac{A}{1-r}R.$$

We have thus constructed a sequence $\underline{y} = \{y_0, \dots, y_N\}$ satisfying

1. $d(f(y_n), x_{n+1}) \leq \frac{1}{1-r}R$
2. $d(x_n, y_n), d(f(y_n), y_{n+1}) \leq \frac{A}{1-r}R$
3. $\underline{y}$ is subordinate to $\mathcal{W}^{cu}$.

It follows by dynamical coherence that if our original pseudo-orbit $\underline{x}$ is subordinate to $\mathcal{W}^{cs}$ then $\underline{y}$ is subordinate to $\mathcal{W}^c$.

Now we apply the same argument to the $\frac{AL}{1-r}R$-pseudo-orbitfor $f^{-1}$, $\underline{y}^{-1} = \{y_N, y_{N-1}, \dots y_0\}$ and get another pseudo-orbit $\underline{z}^{-1} = \{z_N, z_{N-1}, \dots z_0\}$ with the properties

1. $d(f^{-1}z_{n+1}, z_n) \leq \frac{A^2L}{(1-r)^2}R.$
2. $d(z_n, y_n) \leq \frac{A^2L}{(1-r)^2}R$
3. $\underline{z}^{-1}$ is subordinate to $\mathcal{W}^c$.

Finally we end up with a $\left(\frac{AL}{(1-r)}\right)^2 R$-pseudo-orbit$\underline{z} = \{z_0, \dots, z_N\}$ for $f$ subordinate to $\mathcal{W}^c$. Notice that

$$d(x_n, z_n) \leq d(x_n, y_n) + d(z_n, y_n) \leq \frac{A}{1-r}R + \left(\frac{AL}{(1-r)}\right)^2 R = C(f)R.$$

We have thus proved the theorem in the case where the pseudo-orbitis finite. Now suppose that our pseudo-orbit$\underline{x}$ is infinite (for example bi-infinite). The previous argument allows us to find for every $N$ a $C(f)R$-pseudo-orbit $\underline{z}^N$ which
$CR$-shadows the segment $\{x_{-N}, \dots, x_N\}$.

Since $M$ is compact we can find a subsequence $\{N_k\}_k$ such that $z_n^{N_k} \xrightarrow[n\to\infty]{} z_n$. The sequence $\underline{z} = \{z_n\}_n$ is a $C(f)R$-pseudo-orbitwhich $C(f)R$-shadows $\underline{x}$, for numbers $R$ so that $CR < \eta$.

# 4 Proof of Theorem B

The argument is the same used in the proof of Theorem A, using more precise estimates. Since $E^c$ is $\mathcal{C}^1$, the center holonomies are differentiable, and in particular Lipschitz continuous: for $0 < R < \frac{c_{lps}}{2}$ there exists $C_{hol}(R) \geq 1$ so that if $y \in W^c(x, R)$ then

$$z, z' \in D^{su}(x, R) \Rightarrow d(H^c_{y,x}(z), H^c_{y,x}(z')) \leq C_{hol} d(z, z').$$

Similarly for $H^c|\mathcal{W}^s, H^c|\mathcal{W}^u$. Using differentiability of $H^c$ it's not difficult to convince oneself that by taking $x$ and $y$ sufficiently close, and by reducing $R$-accordingly, we can take $C_{hol}(R)$ as close as $1$ as desired when looking at points in the same center-stable or center-unstable leaf (recall that $E^c$ is orthogonal to $E^s$ and $E^u$). In particular one can find $R_0 > 0$ so that $C_{hol} := C_{hol}(R_0)$ verifies $r := \lambda^s C_{hol} < 1$.

On the other hand, due to continuity of the stable and unstable holonomies (and leafwise continuity of the metric in $\mathcal{W}^{cu}, \mathcal{W}^{cs}$) we can guarantee that: for every $0 < R \leq R_0$, given $\epsilon > 0$ there exists $\zeta(\epsilon, R) > 0$ so that

$$\forall x, y \in M, d(x, W^{cu}(y, R)) < \zeta(\epsilon, R) \Rightarrow H^s_{x,y}(W^{cu}(y, R)) \subset W^{cu}(x, R + \epsilon).$$

Fix $0 < R \leq R_0$ and let $0 < \delta_R < R$ so that $\frac{\delta_R}{1-r} < R$.

We now proceed as before: consider first the case when $\underline{x} = \{x_0, \dots, x_N\}$ is a $(R, \delta)$-quasecenter pseudo-orbit, where $0 < \delta \leq \delta_R$, and define the points $y_0, \dots, y_N$ by

- $y_0 = x_0$
- $y_n = [x_n, f(y_{n-1})]$ for $n = 1, \dots, N$.

Note that $d(fx_1, fy_1) < \lambda^s \delta$. Now consider $z_2 = [x_2, fx_1] = W^s(x_2, c_{lps}) \cap W^{cu}(fx_1, c_{lps})$: we have $d(fx_1, z_2) < R$ and $d(x_2, z_2) < \delta$. By dynamical coherence it follows that

$$y_2 = [x_2, fy_1] = H^c_{z_2, fx_1}(fy_1) \therefore d(z_2, y_2) \leq C_{hol} \lambda \delta = r\delta$$

which in turn implies

$$\begin{aligned} d(y_2, x_2) &= \delta + r\delta = \delta(1 + r) \\ d(fy_1, y_2) &\leq E(R, \delta)R \end{aligned}$$

where $E(R, \delta)R$ is the diameter of the smallest center-unstable disk containing the stable projection of $W^{cu}(fx_1, R)$. As explained above, $\lim_{\delta \to 0} E(R, \delta) = 1$.

To argue by induction, we suppose that we have proved that

$$\begin{aligned} d(y_n, x_n) &= \delta(1 + r + \cdots r^{n-1}) \\ d(fy_{n-1}, y_n) &< E(R, \frac{\delta}{1-r})R. \end{aligned}$$

Then

$$d(x_{n+1}, y_{n+1}) \leq r d(y_n, x_n) + d(x_{n+1}, z_{n+1}) < \delta(1 + r + \cdots r^n);$$

hence, if $z_{n+1} = [fx_n, x_{n+1}]$, by hypothesis $z_{n+1} \in W^c(fx_n, R)$ and thus

$$d(fy_n, y_{n+1}) \leq E(R, \delta(1 + r + \cdots r^{n-1}))R \leq E(R, \frac{\delta}{1-r})R$$

Observe that $E'(R, \delta) = E(R, \frac{\delta}{1-r})$ verifies $\lim_{\delta \to 1} E'(R, \delta) = 1$.

From this point on the reader shouldn't have any difficulty to conclude what is claimed in Theorem B, by following the same steps as in Theorem A.

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